ln Before e

by Curtis Silas Helms III

Santa Monica College

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Image via http://en.wikipedia.org/wiki/Logarithm

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Abstract

This article will provide the logarithmic justification for performing exponentiation with real numbers. This line of though is not explored by practioners of exponentation unless they have studies integral calculus. The laws of exponentation for real numbers are also justified using logarithms. Particular focus is paid to e^x due to its wide applicability to continuous growth and decay calculations.

1 Basic Exponentiation

1.1 Notation Assertions

For this article, unless otherwise noted, we asserts

i. Constants a, b are positive real numbers ($a, b \in \mathbb{R}_{>0}$),

ii. Variables n, m range over non-negative integers ($n, m \in \mathbb{I}_{\geq 0}$),

iii. Variables x, y, r range over *all* real numbers ($x, y, r \in \mathbb{R}$),

iv. Variable t ranges over non-negative real numbers ($t \in \mathbb{R}_{\geq 0}$).

1.2 Exponentiation Defined

The exponent in its basic form is a mathematical abbreviation to note the repeated multiplication of a real number by itself. For example, the equations

$$2 \cdot 2 \cdot 2 = 2^3 \text{ and } a \cdot a \cdot a = a^3$$

hold because their right hand sides are abbreviations for the left hand side. In the basic form, only integers can serve as exponents. Consequently, any non-zero number can serve as an exponential base.

Definition 1.

$$a^n \equiv \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$$

When the exponent is negative, the expression serves as the denominator of a fraction with one as the numerator.

Definition 2.

$$a^{-n} \equiv \frac{1}{a^n}$$

There exists a special case with respect to exponentiation:

$$\text{For all } a \neq 0, \quad a^0 = 1 \tag{1}$$

1.3 The Critical Restriction

At this point in our knowledge of exponentiation, we must observe the critical restriction that *only integers can serve as exponents.* Although, with today's calculators and math software a pratictioner can utilize floating point, irrational, and transendental numbers as exponents, we have not yet developed a framework that explains what these devices are doing to yield an answer. Therefore, until we acquire this knowledge, one should employ caution and restrict exponents to integers. That said, we can observe some common properties for exponetiation that are widely known.

$$a^n a^m = a^{n+m} \tag{2}$$

$$(a^n)^m = a^{nm} \tag{3}$$

$$(ab)^n = a^n b^n \tag{4}$$

These equations are simply algebraic manipulations of the multiplication simplifications performed by exponentiation.

1.4 Roots and "Fractional" Exponents

A concept related to exponentation is that of root taking. In root taking the user seeks to discover the number, that when multiplied by itself an integral number of times will equal the base. Formally,

Definition 3. $$y = \sqrt[n]{a} \quad \text{if and only if} \quad y^n = a \ \ , \ a \geq 0$$

Over time certain notations are created to facilitate hand based calculations. One of those notations is the fraction exponent. In reality, it is not an exponent, instead it is what computer scientists call *syntactic sugar,* for the the combined operations of exponentation and root taking. Specificially;

Definition 4. $$a^{1/n} \equiv \sqrt[n]{a}$$

Definition 5. $$a^{m/n} \equiv \sqrt[n]{a^m}$$

When the fractional notation is deployed, the fact that exponentation and root taking are inverse operations become obvious.

$$(a^{1/n})^n = a^{n/n} = a^1 = a \tag{5}$$

$$(a^n)^{1/n} = a^{n/n} = a^1 = a \tag{6}$$

Therefore, the integer only restriction for exponents continues to hold, even when fractional notation is used. As the reader will note in definitions (4) and (5), the integer exponent is *always* applied to the base before the root is taken.

To move beyond basic exponentiation we need to focus on one particular base number, it is a mysterious, real number between the integers 2 and 3.

2 INTRODUCING *e*

The number e, known to most as "the base of the natural logarithm" is unique among all numbers, in that it has the ability to characterize continuous growth and decay. Because of this property, e has a special function 'exp(x)' that has been designated to enable computer programs to implement it efficiently. The first property you need to know about e is its location on the number line.

$$2 < e < 3 \tag{7}$$

The important aspect of (7) to remember is that e is a constant greater than zero. This fact implies that properties (1) to (4) are true for e, even though we do not know its exact value.

2.1 CALCULATING EXP

The primary utility for using e as an exponent base is its ability to enable us to perform continuous growth and decay calulations. The basic concept is to imagine how would we calculate the growth rate of a quantity that is doubling over a over a set period of time.

$$A = A_0(1 + r) \text{ where } r = 1 \text{ yields } A = 2A_0. \tag{8}$$

Next, consider if we were to divide our time period into n sub time periods. This alteration would transform statement (8) into

$$A = A_0\left(1 + \frac{1}{n}\right)^n \text{ where } 2A_0 < A < 3A_0 \text{ and } n > 0. \tag{9}$$

Consequently, when we take n to larger and larger values the value of $(1 + 1/n)^n$ approaches the numerical value for e, to any degree of accuracy.

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.718281828... \tag{10}$$

The floating point number in (10) was calculated using another method that we will cover later. For most calculation $e \approx 2.71828$ is all you need to remember.

The following is Jacob Bernoulli's 1683 definition[1] for the exponential function,

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \tag{11}$$

The aspect about e that is notable is the fact we can use its value as the exponential base to determine growth and decay values for any rate r and time period t. The following formula details how this calculation is performed.

$$A = A_0 e^{rt} \quad \text{where} \quad r \in \mathbb{R} \ \text{and} \ t \in \mathbb{R}_{>0}. \tag{12}$$

This statment will empower the user to determine the final quantity for any positive real multiple of the base time period. We calculate decay by asserting $r < 0$ and growth when $r > 0$. Consequently, when e is deployed as the exponential base, a continuous rate of change is simple to determine.

While statement (11) does express an algebraic form for e^x , this form is not useful for normal usage. Another form of this expression will be introduced later in this article that will enable, person or computer, to compute e^x to any desired degree of accuracy.

The other consideration with respect to (12) is that often we know A and A_0. Thus given r want to determine t or vice versa. As of this point, we have no method for performing this type of analysis. Indeed, the analytical power of equation (12) is not accessable except for trivial calculations, due, in part, to the *integer exponent* constraint. Tragically, the extraodinary utility of equation (12) is sealed behind a mathematical shield.

2.2 The delima of non-integer exponents

The capability of e^x to perform wide ranging growth and decay calculations is the primary reason that professionals in the areas of business and science utilize it. Yet, until we perform our analysis, virtually all e^x calculations are mathematically unjustified. We have seen that e^x will enable very useful calculations, but only if x can range over all real numbers. At this point in our knowledge, we can only allow x to range over integers. Consequently, we can utilize e^n for only the most trivial types of calculation. Clearly such a powerful analytical tool was not designed to be limited to such elementary tasks. And this intuition is correct. Yet, we can only liberate e from its integer exponent confines after we perform a comprehensive analysis of its mathematical ancestor, the natural logarithm.

3 True Dependency Between $\ln x$ and e^x

For many that use the equations $y = e^x$ and $A = A_0 e^{rt}$, how they are able to work their magic is not a matter of concern. Yet, as we have detailed, these equations will do not hold unless further analysis is performed to justify their results. The conceptual foundation on which to build a robust exponentation structure is that of the *natural logarithm*. This function was discovered by John Napier in 1614[2]. Johannes Kepler used it to develop his *three laws of planetary motion*. Afterward, Issac Newton developed the *theory of gravity* from the output of Kepler's equations.

1. http://en.wikipedia.org/wiki/Logarithm
2. http://en.wikipedia.org/wiki/Logarithm

The natural logarithm's utility for early scientists resided in the fact that its output of a product is equal to the sum of its individual outputs. This sentence will be formally defined below. Consequently, all multiplication problems are transformed to addition problems when you have access the logarithm/antilogarithm tables.

3.1 Natural Logarithm Function Defined

Definition 6.

$$\ln x \equiv \int_1^x \frac{dt}{t} \ , x > 0 \tag{13}$$

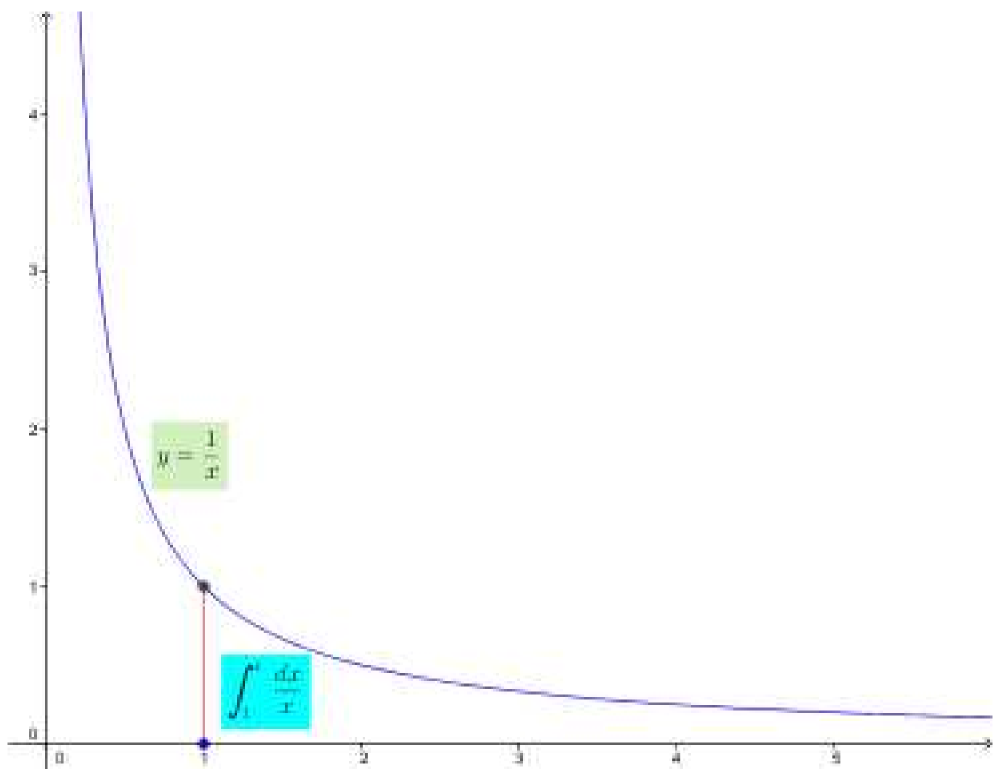

Figure 1.

This definition will enable a large variety of capabilities, when combined with other calculus laws, for the ln x and e^x functions. In your computer spreadsheet programs, the natural logarithm is represented as LN($\cdot$). The key properties you will need to remember regarding this function are:

$$\text{LN}: \mathbb{R}_{>0} \rightarrow \mathbb{R} \tag{14}$$

$$\text{LN}(x \cdot y) = \text{LN}(x) + \text{LN}(y) \tag{15}$$

3.2 Properties of Natural Logarithm

With definition (6) we can determine, by examination, certain properties of the natural logarithim function (ln x). An important propery can be determined from the definition. Namely, $\ln x$ accepts only positive real numbers as inputs. The type signature for the natural logarithm is that of a function with a domain of positive real numbers and all real numbers as its codomain.

Proposition 7.

$$\ln : \mathbb{R}_{>0} \rightarrow \mathbb{R}$$

Proof. By axiom, $\frac{1}{x}$ is undefined at $x = 0$. Hence, the domain of integration for $\ln x$ can only include real numbers greater than 0. Consequently, the domain for $\ln x$ only contains positive real numbers. □

Proposition 8.

$$\ln 1 = 0$$

The only root of $\ln x$ is 1. This result is used often in calculations that involve logarithms.

Proof. By the rules for the definite integral

$$\int_1^1 \frac{dx}{x} = 0.$$

□

Proposition 9.

$$0 < a < 1 \ \textit{implies} \ (\ln a) < 0$$

When $0 < a < 1$, then $\ln a$ measures the area under the curve $1/x$ from a to 1. The definite integral denotes the continuous sum of the area between a function and the abscissa (x-axis), within an interval $x \in [\alpha \text{ to } \beta]$.

$$\int_\alpha^\beta f(x)\, dx = -\int_\beta^\alpha f(x)\, dx, \text{ where } \alpha < \beta \tag{16}$$

Consequently, we must rearrange the order of our interval bounderies to conform to the natural logarithim definition. It requires that the number 1 serve as the lower bound for the domain of integration.

Proof.

$$\begin{aligned}\ln a &= \int_a^1 \frac{dx}{x}\\ &= -\int_1^a \frac{dx}{x}\end{aligned}$$

Since $\frac{1}{x}$ is always positive for $x>0$, then $(\ln a) < 0$ when $0<a<1$. □

Proposition 10. $$\ln(a\,b) = \ln a + \ln b$$

This proposition is what made logarithms the rage among scientists until the advent of low cost calculators. If you ever witnessed anyone using a *slide rule*, you are seeing the direct use of logarithm calculations.

Proof.

$$\begin{aligned}\ln(ab) &= \int_1^{ab} \frac{dx}{x}\\ &= \int_1^a \frac{dx}{x} + \int_a^{ab} \frac{dx}{x}\\ &\quad \{u = x/a\}\\ &= \int_1^a \frac{dx}{x} + \int_{u(a)}^{u(ab)} \frac{a}{a\,u}\,du\\ &= \int_1^a \frac{dx}{x} + \int_1^b \frac{du}{u}\\ &= \ln a + \ln b\end{aligned}$$

□

Proposition 11. $$\ln a^b = b \ln a$$

With the emergence of handheld electronic calculators, slide rules are no longer needed to perform multiplication operations. Logarithms now serves as the means to *lift* the exponent from $y = e^x$ calculations. This proposition is the mathmatical justification for such an operation.

Proof.

$$\begin{aligned} \ln a^b &= \int_1^{a^b} \frac{dx}{x} \\ & \quad \{t = \sqrt[b]{x}\} \\ &= \int_1^{t(a^b)} b\,\frac{t^{(b-1)}}{t^b}\,dt \\ &= b\int_1^{a} \frac{dt}{t} \\ &= b\ln a \end{aligned}$$

□

Proposition 12. $$\frac{d}{dx}\ln x = \frac{1}{x} \text{ and } y = \ln x \text{ implies } dy = \frac{dx}{x}$$

This proposition is one of the first practical results from the Fundamental Theorem of Calculus.

Proof.

$$\begin{aligned} \ln x &\equiv \int_1^{x} \frac{dt}{t}, \qquad x > 0 \\ \frac{d}{dx}\ln x &= \frac{d}{dx}\left(\int_1^{x} \frac{dt}{t}\right) \\ & \quad \{\text{By FTC Part 1}\} \\ &= \frac{1}{x} \end{aligned}$$

□

Lemma 13. $$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

Proposition 14. $$y = a^x \;\; implies \;\; \frac{dy}{dx} = a^x \ln a$$

The proposition will serve as the basis for the amazing quality that the rate of change for e^x is always e^x. By now you should have noticed a pattern. Namely, all of the properties for which e^x have as their mathematical justification the properties of the natural logarithm.

Proof.

$$\begin{aligned} y &= a^x \\ \ln y &= x \ln a \\ \frac{dy}{y} &= \ln a \, dx \\ \frac{dy}{dx} &= a^x \ln a \end{aligned}$$

□

Proposition 15. $$y = \log_a b \;\; implies \;\; y = \frac{\ln b}{\ln a}$$

This proposition is not held in high enough regard. It enables the user to calculate any logarithm for any positive base number. Thus, $\log_2 x$ and $\log_{10} x$ calculations are possible for any real number x. Of course, now we use calculators to obtain our results. Hence, this proposition finds its utility when deployed for symbolic manipulation.

Proof.

$$\begin{aligned} y &= \log_a b \\ a^y &= b \\ y \ln a &= \ln b \\ y &= \frac{\ln b}{\ln a} \end{aligned}$$

□

Proposition 16. $$y = x^r \quad \textit{implies} \quad \frac{dy}{dx} = r x^{r-1} \ , \ r \in \mathbb{R}_{\neq 0}$$

We added this proposition for the calculus student. It enables the application of the power law to real exponents.

Proof.

$$\begin{aligned} y &= x^r \\ \ln y &= r \ln x \\ \frac{dy}{y} &= r\,\frac{dx}{x} \\ \frac{dy}{dx} &= r\frac{x^r}{x} \\ &= r x^{r-1} \end{aligned}$$

□

Henceforth, expressions 'ln a', 'ln b' are just another way to express real numbers. In advanced mathematics, the floating point values of the such denotations are not calculated. The term used to describe these expressions is "algebraic numbers". Therefore, unless the actual value is required, an expression such as '$\ln 10$' would be the final answer given in most situations. Although, we do have the *guarantee of the definite integral* so that for any $a > 0$, we can obtain a floating point value for '$\ln a$'.

3.3 Exponential Function Defined

The definition for the exponential function is simple, namely

Definition 17. $$e^x \equiv \ln^{-1} x$$

The expression e^x is the most common symbol that is utilized to denote this function. The definition in everyday language means that if $y = \ln x$ then $x = e^y$. This relation holds for all real numbers y. We will state this property formally in the next section.

In your computer spreadsheet program, the exponential function is represented as $\exp(\cdot)$. The key properties you will need to remember regarding this function are:

$$\text{EXP}\colon \mathbb{R} \to \mathbb{R}_{>0} \tag{17}$$

$$\text{EXP}(x + y) = \text{EXP}(x) \cdot \text{EXP}(y) \tag{18}$$

3.4 Properties of Exponential Function

Proposition 18. $e^x\colon \mathbb{R} \rightarrow \mathbb{R}^+$

Proof. This function exp has been defined as the inverse function of the logarithic function. Consequently, the domain and codomains are reversed. This is a feature of morphisms according to Category Theory. □

Many students would attribute the type signature of the exponential function to the the property $2 < e < 3$. Given this situation, for any real number x, $e^x > 0$. While it is true that this perspective is what is taught to secondary school students, this perspective ignores the transendental nature of e and thus is an incomplete overview of the type signature.

Definition 19. $e \equiv \{x \text{ such that } \ln x = 1\}$ consequently $\ln e = 1$

Axiom 20.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ e^1 &= e \\ e &\approx 2.718281828... \end{aligned}$$

Proposition 21. $\ln e^x = x$

The natural logarithm will enable us to isolate the exponents of e when we seek to perform further analysis that exponent. For example, $A = A_0 e^{rt}$ becomes $\ln\left(\frac{A}{A_0}\right) = rt$.

Proof.

$$\begin{aligned} \ln e^x &= \ln(\ln^{-1} x) \\ &= x \end{aligned}$$

□

Proposition 22. $y = e^x$ *implies* $\frac{dy}{dx} = e^x$

Proof. This is the statement of the amazing property that the exponential function is its own derivative.

$$\begin{aligned} y &= e^x \\ \frac{dy}{dx} &= e^x \ln e \\ \frac{dy}{dx} &= e^x \end{aligned}$$

□

Proposition 23. $y = e^a\, e^b$ *implies* $y = e^{a+b}$

The next few proposition will justify using *real numbers* as exponents. Initially, we will use e as the base number. Later we will expand the base to all positive real numbers.

Proof.

$$\begin{aligned} y &= e^a\, e^b \\ \ln y &= \ln\left(e^a\, e^b\right) \\ &= \ln e^a + \ln e^b \\ &= a \ln e + b \ln e \\ &= a + b \\ y &= e^{a+b} \end{aligned}$$

□

Proposition 24. $y = (e^a)^b$ *implies* $y = e^{ab}$

Proof.

$$\begin{aligned} y &= (e^a)^b \\ \ln y &= b \ln\left(e^a\right) \\ &= a\, b \ln e \\ &= a\, b \\ y &= e^{a\, b} \end{aligned}$$

□

Proposition 25. $y = a^x$ *implies* $y = e^{x \ln a}$, $a > 0$

This proposition is of critical importance, because it justifies the restriction to *positive base numbers* when reals serve as exponents. Henceforth, expressions such as $e^{\sqrt{\pi}}$ will now have a well defined method of analysis.

Proof.

$$\begin{aligned} y &= a^x \\ \ln y &= x \ln a \\ y &= e^{x \ln a} \end{aligned}$$

□

Definition 26.

$$a^x \equiv e^{(x \cdot \ln a)} \quad \text{and} \quad a^{-x} \equiv \frac{1}{a^x}$$

Now that we are able to calculate the value of a positive number raised to a real number exponent, we have stated a formal definition for this form of *advanced* exponentiation. This definition is implemented within your spreadsheet as the function

$$\text{power}(a, x) \;\; \text{where } a > 0\,.$$

Proposition 27. $y = a^x a^y$ *implies* $y = a^{x+y}$

This and the following two propositions will complete the justification for using real exponents for all positive bases. These propositions replicate equations (2) through (4).

Proof.

$$\begin{aligned} y &= a^x a^y \\ \ln y &= x \ln a + y \ln a \\ &= (\ln a)(x + y) \\ y &= (e^{\ln a})^{(x+y)} \\ y &= a^{x+y} \end{aligned}$$

□

Proposition 28. $y = (a^x)^y$ *implies* $y = a^{xy}$

Proof.

$$\begin{aligned} y &= (a^x)^y \\ \ln y &= y \ln a^x \\ &= (\ln a)(xy) \\ y &= (e^{\ln a})^{(xy)} \\ y &= a^{xy} \end{aligned}$$

□

Proposition 29. $y = (ab)^x$ *implies* $y = a^x b^x$

Proof.

$$\begin{aligned} y &= (ab)^x \\ \ln y &= x(\ln a + \ln b) \\ \ln y &= x \ln a + x \ln b \\ y &= e^{(x \ln a + x \ln b)} \\ y &= e^{(x \ln a)} e^{(x \ln b)} \\ & \quad \{e^{x \ln a} = e^{(\ln a)\,x} = (e^{(\ln a)})^x = a^x\} \\ y &= a^x b^x \end{aligned}$$

□

Propositions (27) through (29) justify the contempory use of logarithms of various bases to perform engineering and financial calculations. In these contexts a logarithm is a real exponent of a chosen base number that, when evaluated, will equal the quantity of interest. For example, $\log_2 1024$ is know by every nerd to equal 10. Yet, what is $\log_2 1325$? This is where the propositions just mentioned justify the calcuation of an answer. The actual value is not relevant here. All we need to know is that such an answer exist, and that there are effective methods to calculate it.

4 FUNCTIONS e^x AND $\ln x$ IN FINANCE

4.1 Time Value of Money Calculations with e^{rt}

Anyone that has used a spreadsheet to perform time value of money (TVM) calculations is familiar with the functions FV(·) and PV(·). They perform future value and present value calculations respectively. These function are the backbone of any MBA finance course. Modern spreadsheets now include a host of TVM related functions in addition to FV and PV. Yet, a business person must master the basic two functions before compentently implementing the more advanced derivatives TVM.

The primary mathematical mechanism of the TVM functions is basic exponentiation. Yes, the very functions that we began article. Given, the use of basic exponents, the *interger only constrain* becomes a critical factor. Because of this constraint, these function can only perform discrete time analysis. Namely, the finance analysts must divide their examination period into discrete blocks of time and the TVM function will perform iterative multiplication to grind out an answer. In addition, TVM functions have no inherent "mechanism" to extract growth rate or time period information. Consequently, adjunct functions or the spreadsheets *Solver* capabilities are utilized to extract rate and time parameters.

Similar to the continuous analysis functions we have defined above, the FV and PV function are inverses of one another. Yet, this fact is not apparent without in-depth analysis. Spreadsheet TVM implementations require multiple arguments per function. This fact makes algebraic analysis extremely difficult. As a result, textbooks use a forms of FV and PV that are mathematically tractable and hence do not resemble their spreadsheet counterparts.

In the final analysis, discrete TMV function are appropriate when they accurately model the calculation method for the financial instrument of interest. Continuous compounding yields answers slightly greater and continuous discounting yields answers sightly less than their respective discrete counterparts. Although continuous methods are more flexible and "realistic", many financial products utilize discrete TVM math to determine their pricing and cash flows. Hence, for congruence sake, in many instances one must compromise mathematical elegance for marketplace practicality.

5 EXAMPLES FROM FINANCE

5.1 Continuous Compounding

Continuous compound interest calculations are the most promenent utilizations of e^x in finance. Below are some examples of the type of calculations made possible with the EXP and LN functions. Unless otherwise noted, these examples assert 360 day years and annual rates of interest compounded continuously.

Problem 1. According to FRED the United States Personal Consumption Expenditure Index: Less Food and Energy (JCXFE) in starting in on 2010-Jan-01 was 100.129 and 105.711 on 2013-APR-01. Compute the mean annual inflation rate over this period.

Answer.

We will manipulate this equation to obtain our answers

$$A = A_0 e^{rt}.$$

A) To compute the average annual inflation rate we will use $t = 3.25$ for our time value. The e^x function enables us to calculate an average annual rate regardless of the time boundries selected for examination.

$$\begin{aligned}
105.542 &= 100.129 e^{3.25r} \\
1.0557 &= e^{\,3.25r} \\
\ln 1.0557 &= 3.25r \\
r &= (\ln 1.0557)/3.25 \\
r &= 0.0542/3.25 \\
r &= 0.01668 \\
r &= 1.67\,\% \text{ or } 167\,\text{bps}
\end{aligned}$$

Problem 2. Sam purchased a 100 USD face value zero-coupon note with 215 days for 98.73 USD. What is the annual percentage yield of the note.

Answer.

$$\begin{aligned}
100 &= 98.73 e^{\,r(215/360)} \\
1.0129 &= e^{0.5972 \cdot r} \\
\ln 1.0129 &= 0.5972 \cdot r \\
r &= (\ln 1.0129)/0.5972 \\
r &= 0.0214 \text{ or } 214\,\text{bps}
\end{aligned}$$

Problem 3. Susan is a financial analyst with Jupiter Investments. She has purchased a 10000 USD 10-year bond for 10500 USD. This throws off a bi-annual coupon of 250 USD.
What is the internal rate of return (IRR) for this bond?

Answer.

The IRR continuous discount rate that will equate all of the bond's future cash flows to the its purchase price.

$$10500 = 10000 e^{-(10r)} + \sum_{n=1}^{20} 250 e^{\,-(r/2)n}$$

Since the discounted principle is added to the discounted coupons, we can not use '$\ln x$' to develop an closed form answer. Consequently, we must utilize a spreadsheet solver. We submit to the solver the problem:

For our equation, find $r>0$ such that LHS = RHS.

From the spreadsheet we obtain the answer

$$r = \text{IRR} = 433\,\text{bps}$$

Problem 4. Raymond would like to grow his investment 10 times within 10 years. What is the minimum rate of interest he could accept on a 10 year instrument to accomplish this task.

$$\begin{aligned} A &= A_0\, e^{rt} \\ 10 &= e^{10r} \\ \ln 10 &= 10r \\ r &= \frac{\ln 10}{10} \\ \\ r &= 23.03\% \;\; \text{or } 2303 \text{ bps} \end{aligned}$$

Problem 5. Nancy, Raymond's banker, has informed him that 500 bps/year (5%) is the most he can reasonably expect to earn on any long term instrument in this economy. Consequently, how long will it take for Raymond to receive his nonimal ten fold return.

$$\begin{aligned} A &= A_0\, e^{rt} \\ 10 &= e^{(500/10000)\,t} \\ 10 &= e^{0.050t} \\ \ln 10 &= 0.050t \\ t &= \frac{\ln 10}{0.050} \\ \\ t &= 46.05 \text{ years} \end{aligned}$$

Problem 6. Behemoth Financial redeemed a 1M USD face value instrument that was purchased two years ago for 850000 USD. Assume the Core CPI rose from 100 to 106 during the same two year period. What was Behemoth's real rate of return for this instrument over the two year period.

First, we will calculate the nominal rate of return (annual)

$$\begin{aligned}
1000000 &= 850000\, e^{2\,R_{\text{Nom}}} \\
\frac{1.00}{0.85} &= e^{2\,R_{\text{Nom}}} \\
\ln 1.176 &= 2\,R_{\text{Nom}} \\
(\ln 1.176)/2 &= R_{\text{Nom}} \\
R_{\text{Nom}} &= 813 \text{ bps}
\end{aligned}$$

Next, calculate the average annual inflation rate.

$$\begin{aligned}
1.06 &= e^{2\,R_{\text{Inf}}} \\
\ln 1.06 &= 2\,R_{\text{Inf}} \\
(\ln 1.06)/2 &= R_{\text{Inf}} \\
R_{\text{Inf}} &= 291 \text{ bps}
\end{aligned}$$

Lastly, determine the real annual rate of return by simple subtraction.

$$\begin{aligned}
R_{\text{Real}} &= R_{\text{Nom}} - R_{\text{Inf}} \\
&= (813 - 291)\,\text{bps} \\
&= 522\,\text{bps}
\end{aligned}$$

6 Conclusion

This article had provided the reader with a secure mathematical foundation to perform exponential and logarithmic calculations with real numbers. It has highlighted the positive number constraint when performing exponentiation with real exponents. The article also made explicit the fact to only positive numbers can serve as arguments for the natural logarithm function.