Development of a valuation tool for American options paying discrete dividends

Better understand and price American options on underlyings paying discrete dividends. The other goal is to generate and calibrate a model as fit as possible, avoiding arbitrability of the volatility surface.
## Contents

1. **ECONOMIC AND SCIENTIFIC CONTEXT**  
2. **THE PROJECT**  
   2.1. Main purpose  
   2.2. Scientific and technical uncertainties, technological obstacles and constraints  
3. **STATE OF THE ART**  
   3.1. Black & Scholes Model  
   3.2. Evaluation of American options  
      3.2.1 Binomial Tree  
      3.2.2 Monte-Carlo simulations  
      3.2.3 Finite difference numerical method  
   3.3 Methods to account for discrete dividend-paying stocks.  
   3.4 Volatility modeling  
      3.4.1 Model SABR  
      3.4.2 IVR Model  
      3.4.3 Local volatility  
4. **R&D WORK CARRIED OUT**  
   4.2 Developments made  
      4.2.1 Calibration model of a local volatility surface for an underlying with discrete dividends  
      4.2.2 Valuation Model for American Equity Options with Dividends  
5. **CONCLUSIONS**  
6. **BIBLIOGRAPHIC RESEARCH AND TECHNOLOGICAL WATCH**  
7. **ANNEXES**  

---

2.  
3.  
4.  
5.  
6.  
7.
1. ECONOMIC AND SCIENTIFIC CONTEXT

Founded in October 2012, with initial focus on trading dividend derivatives, Melanion launched a volatility fund in January 2017 to diversify its activity and seize new investment opportunities.

To ensure high profitability for its clients, Melanion Capital is constantly looking for new trading strategies in niche markets. Moreover, for us to respond and adapt as quickly as possible, optimizing and improving our management tools remains our top priority. Currently, our 4 main areas of R&D development are on dividend derivatives, volatility strategies, cryptocurrencies and sports betting.

2. THE PROJECT

From a technical point of view, the aim of this project is to better understand the evolution of the American Options prices on underlyings paying discrete dividends. Thus, to evaluate this type of financial product, it will be necessary to be able to create a tool that will complement our existing European options evaluation tools, as well as to build local volatility surfaces for identification of arbitrage opportunities. In particular, this will allow us to evaluate volatility swaps on equities. In what follows, we break down these ambitions into technical objectives to give readers a structured view of the project objectives described in this document.

2.1. Main purpose

Our research project is therefore part of the development of the volatility trading activity and revolves around two main axes. The first is the construction of a local volatility surface allowing the valuation of derivatives without arbitrage, taking into account discrete dividends. The second is the calibration of a model for pricing American stock options paying discrete dividends.

In the following sections, we begin by introducing the uncertainties and other obstacles to achieve the above objectives. Then, we will study the solutions available in the scientific and financial literature and we will make a systematic review before introducing our work.

2.2. Scientific and technical uncertainties, technological obstacles and constraints
The financial literature has dealt extensively with the valuation of derivatives on underlying paying continuous or proportional dividends, but little with discrete dividends. Strictly speaking, there are no closed formulas for valuing an American option, the addition of discrete dividends further complicates their valuation.

The first uncertainty is the absence of a stable valuation model for US options with consideration of discrete dividends. This is partly because the usual models, like the Black and Scholes model - described below - do not take into account the effect of discrete dividends. This is because in the Black and Scholes model stock dividends are paid continuously, but in reality, dividends are paid discretely. The risks that arise in this context are mainly the potential losses resulting from an incorrect valuation and an ineffective hedging strategy. Most of the options traded on stocks are American style. The effect of dividends on the price of American options is different from that of European options: dividend payments have an impact not only on the price of the option, but also on the optimal exercise strategy. Another complicating element that comes on top of the discrete nature of the dividends that we want to model.

The second uncertainty that we have encountered is the possibility of creating a stable and regular local volatility surface, taking into account the detachment of discrete dividends during the life of the option. We address both of these issues in this paper. The discrete nature of the dividend must therefore be explicitly modeled.

3. STATE OF THE ART

Since 1970, derivative products have gained a prominent place in financial exchanges. Indeed, access to the latter was greatly facilitated by the creation of the first organized market, in Chicago in 1973, which had the effect of reducing the counterparty risk on various transactions.
3.1. Black & Scholes Model

The central question here is obviously that of the price on which the two parties to the contract must be able to agree. Black, Scholes and Merton (1973) define the price of a derivative as "the price of its hedge". Their model is based on many "simplifying" assumptions, obstacles that we will have to overcome in order to properly value a European option whose underlying pays dividends.

The Black and Scholes formula is a closed formula and defined as follows:

- The value of a Call \( C_t \) with strike \( K \) at time \( t \) and maturity \( T \) is given by:

\[
C_t = S_t e^{-qt} \mathcal{N}(d_1) - Ke^{-rT} \mathcal{N}(d_2)
\]

Where \( S_t \) is the value of the underlying, \( q \) the repo rate, \( r \) the risk-free interest rate, \( \sigma \) the volatility of the underlying and \( \mathcal{N} \) the distribution function of a normal distribution \( \mathcal{N}(0,1) \) with:

\[
\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2} u^2} du
\]

and:

\[
d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}
\]

and:

\[
d_2 = d_1 - \sigma\sqrt{T}
\]

- For a Put, \( P_t \), its value at time \( t \) is given by:

\[
P_t = -S_t e^{-qt} \mathcal{N}(d_1) + Ke^{-rT} \mathcal{N}(d_2)
\]

The Black Scholes model can be used to assess the price of European options, but it naturally cannot be used for American options.
3.2. Evaluation of American options

An American option gives the right to its holder to exercise at any time and until expiration the option he holds:

- The right to buy an underlying for a Call;
- The right to sell an underlying for a Put.

From this definition, we can deduce that an American option cannot be worth less than the equivalent European option:

\[ Option_{American} \geq Option_{European} \]

Furthermore, given the possibility of exercising a US option at any time, its value cannot be less than its intrinsic value, called exercise value. At any date \( t \), an American option with expiration \( T \), strike \( K \) and underlying \( S \) is worth:

\[
Call_{Am}(S,K,t,T) = \max[S - K, \ Call_{Eu}(S,K,t,T)]
\]

\[
Put_{Am}(S,K,t,T) = \max[K - S, \ Put_{Eu}(S,K,t,T)]
\]

Using the Black & Scholes relationship, it is easily identifiable that when interest rates increase, the value of a Put decreases. If a Put is of the American type, then its value cannot be lower than its intrinsic value, which is not the case for a European Put which must comply with the following condition:

\[
\max(Ke^{-rt} - S, 0) \leq P_{Europen} \leq Ke^{-rt} \\
\max(K - S, 0) \leq P_{Americain} \leq K
\]

There may thus be situations where the American Put has a strictly higher value than the European Put. But with the current situation of low or even negative interest rates, the difference between American and European Put has become almost negligible. The Black & Scholes model could therefore theoretically be used to value an American Put under these conditions.

An investor holding an American Call on an underlying that does not pay a dividend will not have an interest in exercising his right because in the first place he would lose the time value of the option: he would have to pay the amount of the strike while he could invest it at the risk-free rate until maturity. Secondly, by exercising the option he would also lose the assurance that it gives him if the
price of the underlying moves against him before maturity. Therefore, the holder will have more
interest in reselling the option rather than exercising it if he wishes to dispose of it.

On the other hand, in the presence of dividends, it may be optimal to exercise the option the day
before the dividend is ex-dividend, in particular when the value of the underlying is sufficiently
high.

Three methods particularly used in the financial industry to value an American option are the
binomial tree, Monte Carlo simulations and the numerical finite difference method.

### 3.2.1 Binomial Tree

The binomial model, developed by Cox, Ross and Rubinstein (1979), is based on the fact that the
value of an option is equal to the expected present value at the risk-free rate of its future payment
(payoff). The binomial tree represents the different paths that an underlying \( S \) can follow to maturity.
The model breaks down time into sub-periods \( \Delta t \) and assumes that \( S \) can only take two values in each
subsequent sub-period:

- On the Rise: \( S_u = S_0 \times u \);
- On the Decline: \( S_d = S_0 \times d \).

Cox, Ross and Rubinstein propose for the values of \( u \) and \( d \):

\[
 u = e^{\sigma \Delta t} \quad \text{and} \quad d = e^{-\sigma \Delta t} \quad \text{with} \quad d = \frac{1}{u} \quad \text{and} \quad \sigma
\]

representing the volatility of the underlying. Each variation (rise and fall) is associated with a risk-
neutral probability. Since the underlying must yield the risk-free rate \( r \) in expectation, the upward
probability \( p \) is given by:

\[
p \times u + (1 - p) \times d = e^{r \Delta t} \quad \iff \quad p = \frac{e^{r \Delta t} - d}{(u - d)}
\]

The model works by iterative method starting at maturity and going back to each sub-period,
calculating the option as follows:

- At the nodes preceding the maturity date:

\[
\text{Call} = \frac{p \times \max[S_u - K, 0] + (1 - p) \times \max[S_d - K, 0]}{e^{r \Delta t}}
\]

- Anterior knots: \( \text{Call} = \frac{p \times \text{Call}_u + (1 - p) \times \text{Call}_d}{e^{r \Delta t}} \)
For the case of an American Call, it suffices to modify the equation at the previous nodes to take into account the possibility of premature exercises at each sub-period:

\[ \text{Call} = \max \left\{ p \times \text{Call}_u + \frac{(1-p) \times \text{Call}_d}{e^{r\Delta t}}, S - K \right\} \]

The model seems attractive to our problem, but it cannot directly include discrete dividends during the life of the option, because these remove one of the essential characteristics of the binomial tree: it must be recombinant at each node.

The appearance of dividends exponentially explodes the number of nodes to be calculated after their detachment. This model is therefore difficult to use for American stock options paying discrete dividends.

An existing method, called Escrow Method, initially developed by Black in 1975, consists of decreasing the value of the initial underlying \( S \) by the present value of dividends. But this method tends to underestimate the price of Calls, especially for those out of the money. However, in the current period of low interest rates, the objective is precisely to be able to correctly estimate the price of Calls.

In addition, the exercise of an American option depending on the level of the underlying \( S \), and the volatility not being constant in practice over time, the use of a non-constant volatility prevents the tree from being recombinant. Methods exist, like the one developed by Derman et al (1994) called Implied Binomial Tree, but this one happens to be relatively unstable and heavy in terms of computation time, and moreover it does not always give option prices conform to those of the market (Lamya Kermiche -2008).

Finally, the use of a binomial tree taking into account both discrete dividends and non-constant volatility seems to be inadequate and difficult to implement.

**3.2.2 Monte-Carlo simulations**

Monte Carlo simulations allow generating stochastic processes with \( S \) following a geometric Brownian motion:

\[ \frac{dS}{S} = \mu dt + \sigma dW_t \]
With $\mu$ the expected rate of return of the underlying, $\sigma$ its volatility and $W$ a Brownian movement.

Using the Itô process, it is possible to demonstrate that in discrete time, $S$ follows:

$$S_T = S_t e^{\left(\mu - \frac{1}{2} \sigma^2\right)(T-t) + \sigma W_t}$$

with $W_t = \varepsilon \sqrt{\Delta t}$ and $\varepsilon$ a random variable following a normal distribution.

By generating a large number of random processes and discounting the average payoffs at maturity for a European option, it is possible to obtain the value of an option.

This method makes it easy to take into account the dividends distributed $D$ by a share over time by applying:

$$S_T = (S_t - D_t) e^{\left(\mu - \frac{1}{2} \sigma^2\right)(T-t) + \sigma W_t}$$

Moreover, it is possible to transform the constant volatility $\sigma$ by $\sigma(S_t, t)$ depending on the time and the level of the underlying.

The disadvantage of this method is the large number of paths to be simulated in order to converge to the real price of an option. In the case of an American option, it would be necessary at each sub-period $\Delta t$ compare the current value of the option at this date $t$, called the continuation value, with the intrinsic value $(S_t - K)^+$ for a Call or $(K - S_t)^+$ for a Put. In theory, this would mean simulating new paths from $S_t$ to obtain the continuation value, which itself would be subject to the condition $Call_{Am}(S, K, t, T) = \max[S - K, Call_{Eu}(S, K, t, T)]$. This method is therefore not feasible in practice and would require too much computational effort.

An approximation exists, called the least squares approach (Longstaff, F., & Schwartz, 2001). This method consists in first calculating the payoffs at maturities $V_T$ and approximate their values by a function of the type:

$$V_T = a + b S_{t-\Delta t} + c S_{t-\Delta t}^2$$

The coefficients $a$, $b$ and $c$ are calculated by the least squares method. By proceeding with an iterative backward method, we can compare the intrinsic value of each option with its continuation value $V$. However, this method is quite time consuming (Areal, Rodrigues and Armada, 2008). There can also be quite large differences between the true value of the option and the
approximation obtained by a function of this type, especially on the wings. Finally, the function can give negative option values, for example for options that are far out of the money.

### 3.2.3 Finite difference numerical method

The numerical finite difference method seems to be a suitable method for our purpose of valuing an American option. This method is based on the numerical solution by discretization of the Black & Scholes partial differential equation (PDE):

$$
\frac{\partial f}{\partial t} + (r - q)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2} = rf
$$

with $f$ the value of the option, $r$ the risk-free rate and $q$ the repo rate.

There are two methods of solving this equation: the implied method and the explicit method. To value an American option, we need to compare the exercise value with the continuation value. For this purpose, the explicit method is the most practical method to solve this problem.

First of all the construction of a grid is necessary, containing the values of the spot $S$ from $S_{\text{min}}$ to $S_{\text{max}}$ distributed by sub-period $\Delta t$ until maturity.

We pose:

- $\Delta S = \frac{S_{\text{max}} - S_{\text{min}}}{\text{Nombre}_{\Delta S}}$ with $\text{Nombre}_{\Delta S}$ determined upstream;
- $\text{Nombre}_{\Delta t} = \text{Nombre Jours Ouvrés} \times \text{Facteur}_{\Delta t}$;
- $\Delta t = \frac{1}{\text{Facteur}_{\Delta t}}\frac{255}{\text{Facteur}_{\Delta t}}$ with $\text{Facteur}_{\Delta t}$ determined upstream.
Assuming a $\Delta t$ very small, it is possible to approximate the derivatives of the PDE as follows:

$$\frac{\partial f_{i,j}}{\partial S} \approx \frac{f_{i+1,j+1} - f_{i-1,j+1}}{2\Delta S} \quad \frac{\partial^2 f_{i,j}}{\partial S^2} \approx \frac{f_{i+1,j+1} + f_{i-1,j+1} - 2f_{i,j+1}}{(\Delta S)^2} \quad \frac{\partial f}{\partial t} \approx \frac{f_{i,j+1} - f_{i,j}}{\Delta t}$$

Replacing the derivatives of the PDE by their approximation and rearranging the equation, we obtain:

$$a_if_{i-1,j+1} + b_if_{i,j+1} + c_if_{i+1,j+1} = f_{i,j} \quad (1)$$

With:

$$a_i = \frac{1}{1 + r\Delta t} \left[-\frac{1}{2} (r - q)i\Delta t + \frac{1}{2} \sigma^2 i^2 \Delta t \right]$$

$$b_i = \frac{1}{1 + r\Delta t} \left[1 - \sigma^2 i^2 \Delta t \right]$$

$$c_i = \frac{1}{1 + r\Delta t} \left[\frac{1}{2} (r - q)i\Delta t + \frac{1}{2} \sigma^2 i^2 \Delta t \right]$$

For an American Call, we can calculate the bounds of our grid:

- At maturity: $f_{i,j_{\max}} = (i\Delta S - K)^+$;
- Upper terminal: $f_{i_{\max},j} = i_{\max} \Delta S - K$;
- Lower terminal: $f_{0,j} = 0$.

Knowing all the blue areas on the grid below, we can solve equation (1) starting at $T_{\max} - \Delta t$ and $(i_{\max} - 1) \Delta S$. Using a backward iterative method, we can calculate the price of an option at $t_0$. For an American option, we calculate at each node:

$$f_{i,j} = \text{Max}[\{a_if_{i-1,j+1} + b_if_{i,j+1} + c_if_{i+1,j+1}\}, i\Delta S - K]$$
Although attractive, this method has two drawbacks. The first is the lack of stability associated with the use of the explicit method, unlike the implicit method. The second is the integration of dividends in the calculation. Indeed, we cannot simply decrease each spot in the grid by the amount of dividends when they are paid.

3.3 Methods to account for discrete dividend-paying stocks.

Here we briefly review the scientific and financial literature on modifications to stock price models to account for discrete dividend payments. Merton (1973) analyzed the effect of discrete dividends in American call options. Roll. 1977], [Geske. 1979] and [Barone-Adesi et al. 1986] worked on the problem of analytical approximations of American options. Hull.1989] in the first edition of his book, 1989, establishes what was to be the most widely used method for dealing with discrete dividends. The method consists of subtracting from the current price of the asset the net present value of all dividends that will be paid during the life of the option. Its popularity and acceptance is due to the fact that it would preserve the continuity of the option price until the dividend payment date and that it could cope with multiple dividends. This method tends to understate the value of the option, especially on the wings.

At the other end of the spectrum, [Musiela et al. 1997] propose a model that adds to the strike price the future value at maturity of all dividends paid during the life of the option. It can be shown that this formulation is only accurate if the dividend payment occurs just before the option expires. To balance the latter two methods, [Bos et al. 2002] devise a method that divides dividends into "near"
and “far” and subtracts the "near" dividends from the stock price and adds the "far" dividends to the Strike. This method performs better than the previous one, but it is not exact, especially for very out-of-the-money options.

A method that accounts for continuous geometric Brownian motion with jumps at dividend payment dates is analyzed in detail by [Wilmott. 1998] using numerical methods. Haug et al. 2003] review the performance of existing methods and pay particular attention to the negative price problem that arises in the context of the jump model and propose a numerical scheme in quadrature. These methods do not deal with American option pricing. Recently, [Vellekoop et al. 2006] describe a modification to the binomial tree method to account for discrete dividends and preserve the crucial recombination property, but this does not describe the use of local volatility. However, it is very important to note that this work - and the relatively satisfactory results - are based on two crucial assumptions: the assumption that, between dividend dates, the asset follows log-normal dynamics, and that the same dynamics is used to value all derivatives.

The shortcomings of these methods are that they either underestimate option values outside the currency, cannot take into account local volatility, or are too time-consuming to calculate in practice. These papers do not meet our conditions: the use of local volatility, the integration of discrete dividends in the valuation, the consideration of premature exercise for an American option and a fast computation time.

3.4 Volatility modeling

3.4.1 Model SABR

The only parameter that is not explicitly given by the market in the Black & Scholes model is volatility. Each market participant estimates a volatility, which is then used to obtain the price of an option.

Many practitioners use the SABR model, developed by Patrick S. Hagan, Deep Kumar, Andrew Lesniewski, and Diana Woodward, to estimate this volatility and generate non-arbitrable volatility surfaces. It is based on the modeling of stochastic volatility.

The advantage of this model is that it has only 4 parameters to estimate for each option maturity:

- \( \sigma \): level of volatility of the currency;
- \( \alpha \): convexity of volatility;
- \( \rho \): impacts the skew;
• β This also affects the skew and the type of price distribution (more or less lognormal).

In this model, the forward price \( F \) of a financial asset follows the following stochastic motion:

\[
\begin{align*}
  \text{d}F_t &= \sigma_t F_t \text{d}W^1_t \\
  \text{d}\sigma_t &= \alpha \sigma_t \text{d}W^2_t \\
  \text{d}W^1_t \text{d}W^2_t &= \rho \text{d}t
\end{align*}
\]

with \( W^1 \) and \( W^2 \) two random processes following a normal distribution and \( \rho \) their correlation.

Solving this equation and using the Black & Scholes model, the following equation is obtained:

\[
\sigma_{SABR}(K,F) = \frac{\sigma}{(FK)^{\frac{1-\beta}{2}} \left( 1 + \frac{(1-\beta)^2}{24} \ln \left( \frac{F}{K} \right)^2 + \frac{(1-\beta)^4}{1920} \ln \left( \frac{F}{K} \right)^4 + \ldots \right) \left( x(z) \right)}
\]

\[
\times \left[ 1 + \frac{(1-\beta)^2}{24} \frac{\sigma^2}{(FK)^{1-\beta}} + \frac{1}{4} \frac{\rho \beta \alpha}{(FK)^{1-\beta}/2} + \frac{2 - 3 \rho^2}{24} \alpha^2 (T - t) + \ldots \right]
\]

With:

\[
\begin{align*}
  z &= \frac{\alpha}{\sigma} (FK)^{\frac{1-\beta}{2}} \ln \left( \frac{F}{K} \right) \\
  x(z) &= \ln \left( \frac{\sqrt{1 - 2 \rho z} + z + \rho}{1 - \rho} \right)
\end{align*}
\]

The interest of this model is that it allows us to obtain the implied volatility of market prices using an analytical formula. It also allows to capture the dynamics of the volatility smile for each option maturity and strike.

On the other hand, this model has certain disadvantages. The parameters must be calculated for each maturity and their value may be unstable over time, making it difficult and imprecise to smooth the volatility surface over time and strike.

### 3.4.2 IVR Model

Another popular model in the financial community is the Stochastic Volatility Inspired (SVI) model, proposed by Gatheral in 2004. It proposes a parameterization of implied volatility by the following model with 5 parameters to estimate the total variance \( W \):

\[
w_{SVI}(X,t) = a + \sigma \times \left[ \frac{\rho(X - m) + \sqrt{(X - m)^2 + \sigma^2}}{1 - \rho} \right]
\]
With \( X \) representing the log-moneyness of the option: \( X = \ln \left( \frac{K}{F} \right) \). The total variance is related to the implied volatility by: \( w(X,t) = \sigma(X,t)^2 \times t \).

The following conditions must be met:

- \( a \in \mathbb{R} \) with \( a \) directly impacting the total variance level vertically;
- \( b \geq 0 \) with \( b \) impacting the slope of the total variant on the wings;
- \( -1 < \rho < 1 \) with \( \rho \) impacting the skew;
- \( m \in \mathbb{R} \) with \( m \) having an impact on the smile;
- \( \sigma \) impacting the curvature of the smile at the currency.

To calibrate the model, the parameters must be adjusted to reduce the errors between the model and the implied market volatilities as much as possible. This model makes it possible to obtain a volatility surface per maturity very close to that of the market, thanks to the existence of 5 parameters.

To obtain a non-arbitrable volatility surface, the following two conditions must be met:

- Absence of calendar arbitrage: \( \partial_t w(X,t) \geq 0 \); 
- Absence of Butterfly arbitrage ensured by the absence of negative probability density.

A calibration of the SVI model allows to obtain most of the time a model without Butterfly arbitrage. On the other hand, the condition of no calendar arbitrage is difficult to meet, especially on the wings.

A parameterization of the \( \rho \) impacting the skew could be a solution, but the model being very sensitive to each parameter value, it is quite difficult to calibrate its parameters while obtaining a volatility surface close to the market one. The SVI model is therefore an efficient model in terms of calibration with market data, but it can present many arbitrage possibilities, preventing the efficient use of a local volatility afterwards.

### 3.4.3 Local volatility

Local volatility is the instantaneous volatility for a given level of the underlying \( S \) and a specific date in time.

The relationship \( \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \) becomes: \( \frac{dS_t}{S_t} = \mu dt + \sigma(S_t,t) dW_t \). The volatility is thus no longer constant over time.
In 1994, Dupire developed a non-parametric local volatility model, allowing to extract a local volatility surface from the prices of market calls. Any option price must satisfy the following differential equation:

$$\frac{\partial C(K,T)}{\partial t} + (r - q)S_t \frac{\partial C(K,T)}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C(K,T)}{\partial S_t^2} = (r - q)C(K,T)$$

With $C$ the price of a Call, $S$ the value of the underlying asset, $K$ the strike, $T$ the maturity, $r$ the risk-free rate, $q$ the repo rate and $\sigma$ volatility.

This equation is solved in the backward direction. In order to know the volatility at a given date $t$ and an underlying level $S$, Dupire used the following "forward" differential equation of Fokker-Planck:

$$\frac{\partial C(K,T)}{\partial t} = (r - q)(C(K,T) - K \frac{\partial C(K,T)}{\partial K}) + \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 C(K,T)}{\partial K^2}.$$ 

By rearranging it, this equation gives the following local volatility:

$$\sigma^2(K,T,S_0) = \frac{\frac{\partial C(K,T)}{\partial t} - (r - q)\left( C(K,T) - K \frac{\partial C(K,T)}{\partial K} \right)}{\frac{1}{2} K^2 \frac{\partial^2 C(K,T)}{\partial K^2}}$$

In the markets, options are usually quoted in terms of implied volatility. Local volatility can be expressed in terms of the total variance after several successive changes in variables:

$$\sigma^2(K,T,S_0) = \frac{\frac{\partial w}{\partial t}}{\left[ 1 - \frac{1}{w} \frac{\partial w}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} + \frac{1}{4} \left( -\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2} \right) \left( \frac{\partial w}{\partial y} \right)^2 \right]}$$

with $y = \ln \left( \frac{K}{F_r} \right)$.

The early exercise of American options depends on the level of the underlying and the existence of dividends. Using a local volatility that depends on the level of the underlying could help to value American options in a consistent way.

The financial literature does not deal much with the subject of discrete dividends, which is an obstacle in the creation of a local volatility surface. Indeed, when the dividend detachment takes place between two option maturities, the local volatility tends to be no longer smooth and regular,
and it gives inconsistent values especially on the wings. Our objective is therefore to construct a smooth local volatility surface taking into account discrete dividends.

4. R&D WORK CARRIED OUT

4.2 Developments made

4.2.1 Calibration model of a local volatility surface for an underlying with discrete dividends

As we have seen before, local volatility can be used in the evaluation of American options. We will present the method we have developed to obtain a local volatility surface for an underlying with discrete dividends.

It requires the following steps:

1. Extraction of implied volatilities from Puts prices using the Newton-Raphson method;
2. Modeling the implied volatility surface:
   2.1. First parameterization according to the SABR model (Beta fixed) with the Gauss Newton method;
   2.2. Parameterization of Rho (impacting skew) as a function of time, to avoid calendar arbitrage in the generated volatility sheet;
   2.3. Recalibration of implied volatility with SABR keeping Beta fixed and Rho parameterized;
3. Transforming the implied volatility sheet into a local volatility surface:
   3.1. Creation of an implied volatility surface by strike and maturity with the optimized parameters of SABR;
   3.2. Transformation by interpolation of the implied volatility surface obtained into a total variance surface according to moneyness and maturity to take into account dividends;
   3.3. Use of the Dupire equation from the total variance by computing the derivatives required to obtain the local volatility surface.
4.2.1.1 Parameterization of a non-arbitrable implied volatility surface

To explain our method, we have chosen to use as an example the listed options of Linde (BBG code: LIN GY Equity) as of September 26, 2019, having the following dividends:

<table>
<thead>
<tr>
<th>Date</th>
<th>Dividends</th>
</tr>
</thead>
<tbody>
<tr>
<td>12/16/2019</td>
<td>0.78</td>
</tr>
<tr>
<td>3/5/2020</td>
<td>0.84</td>
</tr>
<tr>
<td>5/29/2020</td>
<td>0.84</td>
</tr>
<tr>
<td>8/31/2020</td>
<td>0.84</td>
</tr>
<tr>
<td>12/14/2020</td>
<td>0.84</td>
</tr>
</tbody>
</table>

Surface of implicit market volatilities

We have seen that an American put option on a dividend-paying stock has no incentive to be exercised prematurely in the current low-rate universe. This allows us to obtain an implied volatility from the Put prices. We obtain the implied volatilities of the Puts from our Black & Scholes pricer taking into account the discrete dividends. This pricer was developed by the research team of the Melanion Volatility fund.

To obtain the volatilities we use the Newton-Raphson method. We thus have an algorithm to calculate the implied volatility of an option.

The method uses a Taylor development to first order:

\[ Put_{BS}(\sigma) \approx Put_{BS}(\sigma_0) + \frac{\partial Put_{BS}(\sigma_0)}{\partial \sigma}(\sigma - \sigma_0). \]

Our goal is to:

\[ |Put_{BS}(\sigma) - Put_{Marche}(\sigma_{Objectif})| < \varepsilon \]

with \( \varepsilon \) the maximum error threshold.

By rearranging, we obtain:

\[ \sigma = \sigma_0 - \frac{Put_{BS}(\sigma_0) - Put_{Marche}(\sigma_{Objectif})}{Vega} \]

with \( Vega = \frac{\partial Put_{BS}(\sigma_0)}{\partial \sigma} = \frac{Put_{BS}(\sigma_0 + \Delta \sigma) - Put_{BS}(\sigma_0)}{\Delta \sigma} \).
The previous calculation is repeated until the target is reached. Thanks to this method, we obtain our implied volatility surface according to the market prices of Puts (see Annexes).

**Weight assignment function**

We do not attempt to model the market volatility surface uniformly. Indeed, it is important to model the volatilities at the money before giving importance to the modeling of those on the wings. In other words, the accuracy of the model must be greater at the money than on the wings. For this purpose, we have developed a function giving a weight, coefficient, to each volatility during the parameterization of SABR for a given strike, maturity and future price. We were inspired by the normal distribution function:

\[
\text{Coefficient}(K,F,T) = e^{-0.5\left(\frac{K-F}{F\times\sigma\sqrt{T}}\right)^2}
\]

**This function allows us to give more weight to the volatilities at the money forward** (based on the implied future price). Thus, the further the strike of a given volatility is from the reference future, the less weight we will give it in the model calibration.

Moreover, the function is also time dependent. The closer the maturity, the less useful it is to try to calibrate the volatilities in the wings because they are unlikely to be reached. On the other hand, for longer maturities, it is appropriate to give them more and more importance.

We can see the result on the following graph:
The curves are narrower for short maturities and wider for long maturities. The parameter $\sigma_{\text{Coefficient}}$ parameter allows to choose the amplitude of the curve. The higher the $\sigma_{\text{Coefficient}}$ is high, the more weight the function will give to the values on the wings. After several tests, we have chosen to take $\sigma_{\text{Coefficient}} = 25\%$.

**1st parameterization of the volatility surface**

The next step is to parameterize the SABR model to obtain a modeled surface as close as possible to the market one. To do so, we have to solve a non-linear optimization problem. We seek to minimize the following expression:

$$\sum_{n=1}^{N} \left[ IV_{SABR}(\sigma,\alpha,\beta,\rho,K_n,F_T,T) - IV_{Marche}(K_n,T) \times Coefficient(K_n,F_T,\sigma_{\text{Coefficient}},T) \right]^2$$

To solve this problem, we have developed an algorithm using the Levenberg-Marquardt method.

It is common in the financial industry to set $\beta$ for several reasons:

- Leaving $\beta$ fixed has no impact on the optimization quality of the problem because it has very little effect on the shape of the smile, and thus allows to have only 3 parameters to estimate;
- Fixing $\beta$ close to 1 allows us to have a stochastic log-normal model, and thus a $\sigma$ representing a volatility to the currency.
We have chosen to set $\beta$ to 0.95 in order not to deviate too much from a lognormal distribution. Given the presence of many terms $(1 - \beta)$ in the SABR model, taking $\beta = 1$ would have lost some information.

We first calibrate our SABR model by maturity (with $\beta$ fixed) to the market volatilities adjusted by the calculated weights. We obtain the following values for each maturity:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Future</td>
<td>176.02</td>
<td>175.97</td>
<td>174.98</td>
<td>174.11</td>
<td>172.99</td>
<td>171.89</td>
<td>170.72</td>
</tr>
<tr>
<td>Beta</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
</tr>
<tr>
<td>Sigma</td>
<td>0.24</td>
<td>0.26</td>
<td>0.27</td>
<td>0.28</td>
<td>0.29</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>Alpha</td>
<td>2.46</td>
<td>1.64</td>
<td>1.15</td>
<td>0.81</td>
<td>0.64</td>
<td>0.51</td>
<td>0.50</td>
</tr>
<tr>
<td>Rho</td>
<td>-0.42</td>
<td>-0.45</td>
<td>-0.54</td>
<td>-0.52</td>
<td>-0.54</td>
<td>-0.55</td>
<td>-0.52</td>
</tr>
</tbody>
</table>

2nd parameterization of the volatility surface

In order to obtain a local volatility surface, it is necessary that the implied volatility surface does not present any arbitrage, especially calendar arbitrage. For this, the total variance must be increasing over time. Thanks to the weight function developed, if the market data do not show any calendar arbitrage, the parameterized volatility surface should not show any in the forward currency.

On the other hand, on the wings, arbitrages can appear, especially after dividends have been released. The method we use here to solve this problem is to smooth the skew over time, to avoid erratic values from one maturity to another. Rho $\rho$ in the SABR model directly impacts the skew.
We therefore choose in our model to calibrate $\rho$ with a time-dependent function $t$:

$$\rho(t) = A \times e^{-\frac{t}{B}} + C \times (1 - e^{-\frac{t}{B}})$$

with the condition $B > 0$.

It is enough to find the 3 coefficients $A$, $B$ and $C$ allowing to minimize the differences between $\rho(t)$ and $\rho_t$ calculated previously. This function allows to limit the values of $\rho$ between $A$ and $C$ which will therefore have initial values close to the first and last $\rho$ respectively -0.42 and -0.52 in our example. The coefficient $B$ acts directly on the curvature between the points $A$ and $C$. The closer it is to 0, the more important the curvature will be.

Here are the results of the $\rho$ parameterized:

![Graph of Rho](image)

The optimal coefficients obtained are thus:

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.44</td>
<td>-0.53</td>
<td>-0.57</td>
</tr>
</tbody>
</table>

Once set, we can do a new calibration of our SABR model with $\beta$ fixed and $\rho$ parameterized. There are only 2 parameters to optimize: $\sigma$ and $\alpha$. We obtain a parametrized surface, with a smoothed skew allowing to obtain a local volatility surface.
Here are the results for the June 2020 maturity:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Future</td>
<td>176.02</td>
<td>175.97</td>
<td>174.98</td>
<td>174.11</td>
<td>172.99</td>
<td>171.89</td>
<td>170.72</td>
</tr>
<tr>
<td>Beta</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
</tr>
<tr>
<td>Sigma</td>
<td>0.24</td>
<td>0.26</td>
<td>0.26</td>
<td>0.28</td>
<td>0.29</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>Alpha</td>
<td>2.36</td>
<td>1.60</td>
<td>1.24</td>
<td>0.82</td>
<td>0.64</td>
<td>0.52</td>
<td>0.48</td>
</tr>
<tr>
<td>Rho</td>
<td>-0.45</td>
<td>-0.47</td>
<td>-0.49</td>
<td>-0.52</td>
<td>-0.54</td>
<td>-0.55</td>
<td>-0.56</td>
</tr>
</tbody>
</table>

4.2.1.2 Transformation of an implied volatility surface into local volatility

In order to calculate the local volatility at a point, we need the strike and time derivatives. For the strike derivatives, our SABR model allows us to obtain all the desired values. On the other hand, for desired maturities between 2 existing option maturities, we have to proceed by linear interpolation:

\[ \sigma_{T^*} = \sigma_{T_1} + \frac{T^* - T_1}{T_2 - T_1} \times (\sigma_{T_2} - \sigma_{T_1}) \]

with \( T_1 < T^* < T_2 \) the durations in annual fractions (number of working days / 255).

Since the local volatility function is based on the total variance, we will first transform our implied volatility surface into a total variance surface:

\[ w(K,T) = \sigma_T^2(K,T) \times T \]
Once we have obtained our total variance surface, we can transform it into a local volatility surface. This requires an additional step. As seen before, in the presence of discrete detached dividends, there are discontinuities and this causes inconsistent or zero local volatility values. To overcome this problem, we seek to compute the volatility of an underlying with reinvested dividends which, by definition, will have the same volatility as the considered underlying. To obtain this result, it is possible to modify our total variance surface. We transform our surface by strike and maturity into a surface by moneyness $X = \frac{K}{F_T}$ and maturity. The future $F_T$ allows us to take into account the dividends detached over time. Indeed, after the detachment of a dividend, the value of the future is decreased by the value of the dividend. The moneyness will therefore tend to increase, allowing us to adjust the volatility in our surface. On each row of our matrix will correspond a total variance for a given moneyness.

By interpolation, we can find the total variance corresponding to each moneyness. To do this, we first construct a vector with the different moneyness values, corresponding to $t = 0$ à $X = \frac{K}{S_0}$.

At a date $T$, to find the value of the total variance corresponding to a given moneyness, we must calculate the strike $K^*$ corresponds to this moneyness $X$:

$$\frac{K^*}{F_T} = X$$

$$K^* = F_T \times X$$

So we have:

$$w(X,T) = w(K^*,T) = w(F_T \times X ,T)$$

We can thus easily find the value of the total variance sought by interpolation:

$$w(X,T) = w(K^*,T) = w(K^-,T) + \frac{K^* - K^-}{K^+ - K^-} \times [w(K^+,T) - w(K^-,T)]$$

with $K^- < K^* < K^+$.

This method overcomes the dividend detachment problem by obtaining a correctly calibrated and corrected total variance surface.

To obtain the local volatility, we use the following formula:
Pricing American Options with Discrete Dividends

\[ \sigma^2(y,w,T) = \frac{\partial w}{\partial T} \left[ 1 - \frac{y}{w} \frac{\partial w}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} + \frac{1}{4} \left( 1 - \frac{1}{w} + \frac{y^2}{w^2} \right) \left( \frac{\partial w}{\partial y} \right)^2 \right] \]

with \( y = \ln \left( \frac{K}{F_T} \right) \).

We must first approximate the derivatives in discrete time to be able to calculate them.

**First derivative as a function of time**:

\[
\frac{dw(y,T)}{dt} \approx \frac{w(y,T + dt) - w(y,T - dt)}{2dt} \approx \frac{w(y,T + dt) - w(y,T - dt)}{2 \times \Delta \text{jours}}
\]

**First derivative as a function of log-moneyness**:

\[
\frac{dw(y,T)}{dy} \approx \frac{w(y + dy,T) - w(y - dy,T)}{2dy}
\]

To find \( w(y + dy) \) we have to identify to which moneyness (of which we have the matrix) the log-moneyness \( y + dy \) corresponds. To do this, we pose:

\[ \ln(X') - \ln(X) = dy \]

\[ \Leftrightarrow X' = X \times e^{dy} \]

\[ \Leftrightarrow w(y - dy) = w(X') = w(X \times e^{dy}) \]

\[ \Leftrightarrow w(y - dy) = w(X') = w(X \times e^{-dy}) \]

We can thus calculate the total variance corresponding to each upward and downward derivative from our total variance surface.

**Second derivative as a function of log-moneyness**:

\[
\frac{d^2 w(y,T)}{dy^2} \approx \frac{w(y + dy,T) + w(y - dy,T) - 2w(y,T)}{dy^2}
\]
Thanks to the creation of a total variance matrix according to moneyness, we have succeeded in creating a local volatility surface taking into account dividends thanks to the calculation of futures prices.

4.2.2 Valuation Model for American Equity Options with Dividends

4.2.2.1 Restatement of discrete dividends

In order to properly account for dividend payouts in option pricing, we need to be able to value them.

To do this, we decided to base ourselves on the dividend futures market. These allow us to take a position on the dividends paid by a company in the future. Assuming that the markets are efficient and that there is no arbitrage, these futures represent a correct expectation of dividends.

Nevertheless, it is necessary to make adjustments for dividends paid in several installments. A dividend future is the sum of all dividends paid in one year, between the 3rd Friday of December year N and the 3rd Friday of December year N+1. Thanks to the BDVD function of Bloomberg we can retrieve the average analysts' estimate for each dividend payment.
First, we calculate the average amount of dividends per payout over the future period:

\[ \text{Dividende par versement (Future)} = \frac{\text{Valeur du Future sur dividende}}{\text{Nombre de dividendes BDVD sur la période du Future}} \]

Then, we calculate in the same way with BDVD, always on the period of the future:

\[ \text{Dividende par versement (BDVD)} = \frac{\sum \text{Dividendes BDVD sur la période du Future}}{\text{Nombre de dividendes BDVD sur la période du Future}} \]

This allows us to obtain the adjustment by period:

\[ \text{Ajustement BDVD} = \text{Dividende par versement (Future)} - \text{Dividende par versement (BDVD)} \]

This allows us to restate each BDVD dividend amount, consistent with the futures prices and the number of dividend payments. This method allows us to obtain the dividend payout amounts that we use in option pricing.

### 4.2.2.2 Monte Carlo approximation for American options with discrete dividends

The problem associated with the valuation of American options is the possibility of premature exercise of an option. We have seen that premature exercise can be optimal on the eve of a dividend if the exercise value of a call \((S_t^- - K)^+\) is higher than the continuation value \(V_{\text{Continuation}}\). On the date \(t^-\) the day before the dividend is paid, the value of a Call is equal to:

\[ C(S_t^-, K, t^-, T, \sigma)_{\text{Américain}} = \text{Max}[S_t^- - K, V_{\text{Continuation}}] \]

The exercise value \((S_t^- - K)\) is easily obtained for each generated path and is therefore not a problem. The question was how to calculate the continuation value \(V_{\text{Continuation}}\). The first idea was to generate new paths from the spot \(S_t^-\) but this would make the execution time too long, especially in the presence of several dividends because this step would have to be repeated for each dividend.

We have developed a more time-efficient method by approximating the continuation value. As of \(t^-\) an investor holding an American Call has the choice of exercising his option, or holding a European option with spot \(S_t^-\) strike \(K\) and maturity \(T^* = (T - t^-)\) with \(T\) the maturity of the option. If the underlying pays several dividends during the life of the option, the investor will have the choice each
time between exercising his option the day before the dividend or holding a new option with the same maturity, $t_{n^-}$ of the dividend or to hold a new European option with a spot $S_{t_{n^-}}$ and maturity $T^* = (T - t_{n^-})$.

To speed up the calculation, the European Call will be calculated with our Black & Scholes pricer. The use of this method requires to take a constant volatility over time.

To date $t^-$ we have:

$$V_{\text{Continuation}} = C(S_{t^-}, K, t^- T, \sigma)_{\text{Européen}}$$

$$C(S_{t^-}, K, t^- T, \sigma)_{\text{Américain}} = \text{Max}[S_{t^-} - K, C(S_{t^-}, K, t^- T, \sigma)_{\text{Européen}}]$$

The American Call is equal to the average of the American Calls obtained for each path:

$$C(S, K, t^- T, \sigma)_{\text{Américain}} = \frac{\sum_{n=1}^{N} C(S_{0}, K, t^- T, \sigma)_{\text{Américain}}}{N}$$

Since the value of an American call cannot be less than that of a European call, we have to $t = 0$:

$$C(S_{0}, K, t^- T, \sigma)_{\text{Américain}} = \text{Max}[C(S_{0}, K, t_0 T, \sigma)_{\text{Européen}}, C(S, K, t^- T, \sigma)_{\text{Américain}} \times e^{-r(t^- t_0)}]$$

The value $C(S, K, t^- T, \sigma)_{\text{Américain}}$ must be calculated on the day before each dividend.

Since our method is based on an approximation of the continuation value, errors in evaluation can occur. To correct this, we decided to use the difference control technique. Assuming that the exercise premium added to the price of a European option to obtain the American equivalent is the same between the theoretical prices and our method, we obtain:

$$\Delta_{\text{Prime d’exercice théorique}} = \Delta_{\text{Prime d’exercice calculée avec Monte Carlo}}$$

$$\Leftrightarrow C_{\text{Américain}} - C_{\text{Européen}} = C_{\text{Monte Carlo–Américain}} - C_{\text{Monte Carlo–Européen}}$$

$$\Leftrightarrow C_{\text{Américain}} = C_{\text{Européen}} + [C_{\text{Monte Carlo–Américain}} - C_{\text{Monte Carlo–Européen}}]$$

$$\Leftrightarrow C_{\text{Américain}} = C_{\text{BS–Européen}} + [C_{\text{Monte Carlo–Américain}} - C_{\text{Monte Carlo–Européen}}]$$

By calculating the theoretical value of the European call with our Black & Scholes pricer, we can obtain a model for valuing American calls with discrete dividends.
Contrary to the method developed by Longstaff and Schwartz, this method is even faster because it uses the Black & Scholes pricer rather than a least squares method. The use of pricer also allows to have a continuation value greater than 0, which may not be the case with least squares.

4.2.2.3 Optimal finite difference numerical method

Two problems exist with the use of the finite difference method. The first is the consideration of ex-dividend payments. According to the no-arbitrage principle, the value of a call must respect the following relationship before and after the payment of a dividend at date $t$:

$$C(S,K,t^-,T) = C(S - D,K,t^+,T) \text{ with } t^- < t < t^+.$$ 

The method consists in calculating the value of a call with the finite difference method, as described in part 3, up to the detachment date of a dividend. As soon as we find ourselves at date $t$ where a dividend is detached, an adjustment of the initially calculated prices is necessary. To do this, we adjust each Calls price obtained by interpolation to obtain the value of the corresponding option before the dividend payment.

The Call to the knot $i$ of spot $S = i\Delta S$ before dividend payment $D$ corresponds to the node call $i^*$ and spot $S - D = i^*\Delta S$ after dividend payment:

$$C_{i,t}^- = C_{i^*,t}^+$$

We need to identify which Call $i^*$ it corresponds to using the following method:

$$i^* = \frac{i\Delta S - D}{\Delta S}$$

Calculating the option prices in our grid at discrete intervals, we must interpolate to obtain the option price at the node $i^*$:

$$i_{Arrondi} = \text{ArrondiInférieur}(i^*)$$
With this relationship we can adjust the option price vector at each dividend payment. Once the prices are adjusted, we continue the finite difference method until $t_0$ or to the previous dividend in case of multiple payments. Thanks to this interpolation we have succeeded in taking discrete dividends into account.

The second drawback of the explicit method is its instability. Indeed, negative or inconsistent option values can appear in the middle of the grid, distorting all the previous results necessary to determine the value of the Call at $t_0$. The coefficients $a_i$, $b_i$ and $c_i$ can be respectively interpreted as the discounted probabilities of the underlying declining, remaining stable or rising in the next period. The instability comes from the coefficient $b_i$ which can become negative and generate errors throughout the grid. As a reminder:

$$b_i = \frac{1}{1 + r\Delta t} (1 - \sigma^2 i^2 \Delta t)$$

By posing the following stability condition, we obtain:

$$ (1 - \sigma^2 i^2 \Delta t) > 0 \text{ with } \Delta t = \frac{1}{\text{Facteur}_{\Delta t}}$$

$$\Leftrightarrow \left(1 - \frac{\sigma^2 i^2 \text{Facteur}_{\Delta t}}{255}\right) > 0$$

We seek to find the $\text{Facteur}_{\Delta t}$ optimal for our method to be stable. We find the result:

$$\text{Facteur}_{\Delta t} > \frac{\sigma^2 i_{\text{max}}^2}{255}$$

This relation allows us to obtain a stable numerical method, at the cost of an increase in the number of iterations to be computed because we have established:

$\text{Nombre}_{\Delta t} = \text{Nombre Jour Ouvrée \times Facteur}_{\Delta t}$.
Here are the prices we get for the June 2020 maturity:

<table>
<thead>
<tr>
<th>Maturité</th>
<th>Strike</th>
<th>Bid-Marché</th>
<th>Prix Calculé</th>
<th>Ask-Marché</th>
<th>Prime d’exercice</th>
</tr>
</thead>
<tbody>
<tr>
<td>6/19/2020</td>
<td>140.0</td>
<td>37.70</td>
<td>37.83</td>
<td>38.35</td>
<td>1.30</td>
</tr>
<tr>
<td>6/19/2020</td>
<td>150.0</td>
<td>29.90</td>
<td>29.38</td>
<td>29.70</td>
<td>0.98</td>
</tr>
<tr>
<td>6/19/2020</td>
<td>160.0</td>
<td>21.30</td>
<td>21.82</td>
<td>22.25</td>
<td>0.69</td>
</tr>
<tr>
<td>6/19/2020</td>
<td>170.0</td>
<td>14.95</td>
<td>15.38</td>
<td>15.70</td>
<td>0.46</td>
</tr>
<tr>
<td>6/19/2020</td>
<td>180.0</td>
<td>9.50</td>
<td>10.23</td>
<td>10.55</td>
<td>0.28</td>
</tr>
<tr>
<td>6/19/2020</td>
<td>190.0</td>
<td>6.15</td>
<td>6.40</td>
<td>6.75</td>
<td>0.17</td>
</tr>
<tr>
<td>6/19/2020</td>
<td>200.0</td>
<td>3.62</td>
<td>3.78</td>
<td>4.07</td>
<td>0.09</td>
</tr>
</tbody>
</table>

The results obtained are very satisfactory in terms of calculated prices. Indeed, the objective is to obtain prices calculated as close as possible to the "Mids" prices (average between bid prices and market asks). Here we obtain results close to these Mids values. Moreover, the strike premium, corresponding to the difference between the price of an American and a European option, decreases with the increase of the strike for calls. This is in line with our expectations.

We have thus found two solutions to restate discrete dividends in the valuation of American options and to obtain a stable numerical method to calculate an American option with discrete dividends. Moreover, this method allows the use of a local volatility instead of a constant volatility.
5. **CONCLUSIONS**

5.1. **Results and knowledge gained**

To account for discrete dividend payouts, the main known models apply an adjustment to the spot value of the underlying at the initial date, corresponding to the present value of the dividends, or adjust the spot value on a regular basis during the life of the option. These methods tend to underestimate option values, and also to reveal arbitrage opportunities. First, using the finite difference method, we have succeeded in developing a valuation tool that allows us to calculate both European and American options on stocks paying discrete dividends. This was possible thanks to a stabilization method to obtain reliable results, as well as the use of interpolations respecting the no-arbitrage principle before and after dividend payments. In order to properly use this tool, we had to develop a tool for estimating dividends, an important element in the valuation of American equity calls, as an appendix.

Our research has also allowed us to develop a tool to create non-arbitrable local volatility surfaces, taking into account discrete dividends. To do so, we used a matrix of implied volatility according to the moneyness and not the strike to obtain a local volatility surface. The local volatility obtained will allow us to evaluate products such as volatility swaps, or products whose payments depend on the paths taken by the underlying asset.

5.2. **Prospects for the future**

American options whose payoff depends on the path taken by the underlying can also be valued using local volatility. However, this volatility can become very high (over 100%) on the wings. The problem with our finite difference method is stability. The higher the volatility in the local volatility surface, the more unstable the numerical evaluation will become, requiring a high. This increases the computation time considerably. Another stability method will be needed to use the local volatility with the finite difference method. On the other hand, using Monte Carlo simulations, local volatility can be used to value any type of product, such as volatility swaps on baskets of stocks.

Finally, our tool could be adapted to imply the amount of dividends expected by the options market and could be compared to the price of dividend futures. Arbitrage opportunities could then appear and be exploited.
6. BIBLIOGRAPHIC RESEARCH AND TECHNOLOGICAL WATCH

- Geske, R. (1979). A Note on an Analytical Valuation Formula for Unprotected American Call Options on Stocks With Known Dividends.
No part of this material may be reproduced in any form, or referred to in any other publication without the express written permission of Melanion Capital.

The information provided is for informational purposes only and is subject to change without notice. This report does not constitute, either explicitly or implicitly, any provision of services or products by Melanion Capital, and investors should determine for themselves whether a particular investment management service is suitable for their investment needs. All statements are strictly beliefs and points of view held by Melanion Capital and are not endorsements by Melanion Capital of any company or security or recommendations by Melanion Capital to buy, sell or hold any security. Historical results are not indications of future results. Certain statements may be statements of future expectations and other forward-looking statements that are based on Melanion’s current views and assumptions and involve known and unknown risks and uncertainties that could cause actual results, performance or events to differ materially from those expressed or implied in such statements.

Melanion Capital assumes no obligation to update any forward-looking information contained in this document. Certain information was obtained from sources that Melanion Capital believes to be reliable; however, Melanion Capital does not guarantee the accuracy or completeness of any information obtained from any third party.

Melanion Capital
17 Avenue Georges V
75008 Paris

For info:
Contact@melanion.com