

Math Analysis: Course Study Guide

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Basic Algebra Review

******Before we start looking at harder concepts, it is very important to make sure that you have mastered the basic algebraic rules and concepts because these properties will be applied to the harder concepts.

➤ Arithmetic Operations

- $xy + xz = x(y+z)$
 - As you can see both of the terms on the left side of this equation have 'x' therefore we can factor it out.
- $\frac{x}{y} + \frac{a}{b} = \frac{xb + ay}{yb}$
 - When you are adding or subtracting fractions, both terms must have a common denominator. As you can see here, we multiplied both the numerator and denominator of both terms by the other term's denominator, so they can both be added.
- $x\left(\frac{a}{b}\right) = \frac{ax}{b}$
 - When multiplying and dividing fractions you do not need a common denominator
- $\frac{a}{b} \div \frac{x}{y} = \frac{ay}{bx}$
 - When you are dividing fractions remember the phrase "keep, change, flip". You have to keep the 1st term the same, change the division sign to multiplication, and flip the spots of the numerator and denominator of the 2nd term.

$$\circ \quad \frac{a+x}{b} = \frac{a}{b} + \frac{x}{b}$$

- When you are adding or subtracting 2 terms in the numerator, you can separate them if they share a common denominator

➤ Absolute value

- Absolute value is the distance a number is from zero. If there is a negative number in absolute value brackets it will become positive.
- $|xy| = |x||y|$
- $|-x| = |x|$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

➤ Exponent properties

- $x^a x^b = x^{a+b}$
 - When multiplying 2 terms with exponents you can add their powers
- $(x^a)^b = x^{ab}$
 - When you are raising a term with an exponent to another power you have to multiply the 2 exponents
- $(xy)^a = x^a y^a$
 - In this situation, you are raising the quantity to a power; so you have to distribute the exponents to both terms.
- $x^{-a} = \frac{1}{x^a}$
 - If a term is being raised to a negative exponent, you can get rid of the negative exponent by moving it to the denominator and making the exponent positive.
 - This can also be done if it were the opposite and you had a fraction with a negative exponent in the denominator.
- $x^0 = 1$
 - Any term raised to a power of zero is 1

➤ Properties of Radicals

- $\sqrt[n]{x} = x^{\frac{1}{n}}$
 - When you have a term with a radical, you can get rid of the radical by taking the term out of the radical and raising it to 1 over the power of the radical.
- $\sqrt[x]{\sqrt[y]{a}} = \sqrt[xy]{a}$
 - In this situation, you can multiply the powers of both radicals
- $\sqrt[x]{ab} = \sqrt[x]{a}\sqrt[x]{b}$
 - When you are taking the square root of either a multiplication or division expression, you can distribute the radical to each term.

➤ Distance formula

- The distance formula is used when you have **2 points** and you want to find the **distance** between them.
- If you take the Pythagorean theorem ($a^2 + b^2 = c^2$), and then solve for c you will get the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

➤ Complex Numbers

- $i = \sqrt{-1}$
- $i^2 = -1$
- $\sqrt{-x} = i\sqrt{x}$, $x \geq 0$

➤ Logarithms

- $y = \log_b x \equiv x = b^y$
 - Ex; $\log_5 625 = 4 \equiv 625 = 5^4$
- Natural log: $\ln x = \log_e x$
 - $e = 2.718281828\dots$
- Common log: $\log x = \log_{10} x$
 - Anytime there is no base on log function, you should assume the base is 10

- $\log_b b = 1$
 - When the base and the x value are equivalent that function will equal zero
- $\log_b 1 = 0$
 - Whenever the x value of the log function is 1, regardless of the base, the function equals 0. Why? Because any number raised to the zero power = 1.
- $\log_b b^x = x$
- The domain of all logarithmic functions is always $x > 0$

➤ Factoring Formulas

- Difference of Squares
 - $x^2 - a^2 = (x+a)(x-a)$
- Sum of Cubes
 - $(x+a)(x^2 - ax + a^2)$
- Difference of Cubes
 - $(x-a)(x^2 + ax + a^2)$
- Quadratic formula
 - $ax^2 + bx + c$
 - $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- If $x^2 = a$ then $x = \pm\sqrt{a}$
 - Remember, the square root of a term can be positive and NEGATIVE

➤ Slope Formulas

- Slope-intercept
 - $y = mx + b$
 - $m = \text{slope}$; $b = \text{y-intercept}$
- Slope

$$\blacksquare y_2 - y_1 / x_2 - x_1$$

○ Point-slope

$$\blacksquare y = y_1 + m (x - x_1)$$

➤ Equations of different types of functions

○ Parabola/Quadratic

$$\blacksquare ax^2 + bx + c$$

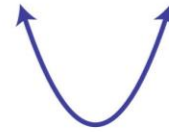
$$\blacksquare y = a (x - h)^2 + k$$

- Opens up if $a > 0$
- Opens down if $a < 0$
- The vertex is (h , k)

Parabola $y = ax^2 + bx + c$

$$a > 0$$

$$a < 0$$



opens upward

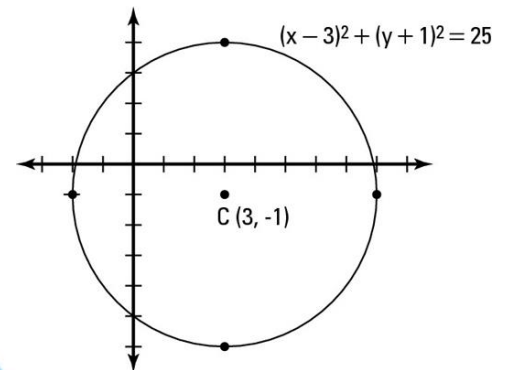


opens downward

○ Circle

$$\blacksquare (x - h)^2 + (y - k)^2 = r^2$$

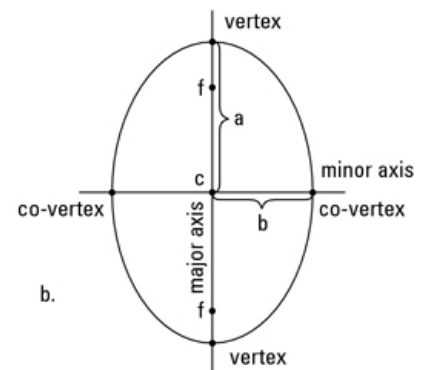
- 'r' represents the radius
- The center of the circle is (h , k)



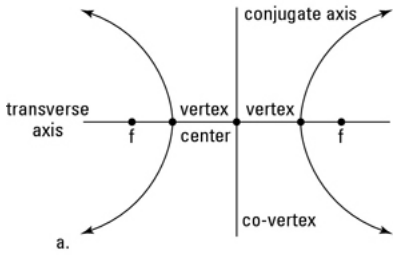
○ Ellipse

$$\blacksquare \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

- Like circles, the center of an ellipse is also (h , k)
- 'a' represents the vertices 'a' units to the left/right
- 'b' represents vertices 'b' units up/down



○ Hyperbola



■ $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

- This equation is very similar to the equation of an ellipse, but the sign in the middle is minus
- The center is still (h , k)
- This equation shows that the graph opens left and right

■ $\frac{(x-h)^2}{b^2} - \frac{(y-k)^2}{a^2} = 1$

- This equation shows that the graph opens up and down

Now we are ready to begin looking at some new content.

Unit 1

Matrix Algebra

● Matrix

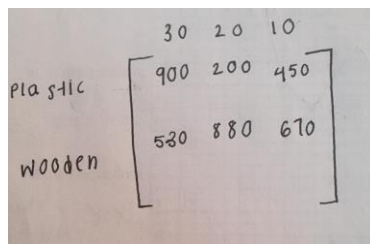
- A matrix is a collection of numbers organized into rows and columns.
- Matrices are referred to as capital letters.

■ Ex; $A_{m \times n}$

	Column1	Column2	Column j	Column n
Row1	a_{11}	a_{12}	a_{1j}	a_{1n}
Row2	a_{21}	a_{22}	a_{2j}	a_{2n}
Row i	a_{i1}	a_{i2}	a_{ij}	a_{in}
Row m	a_{m1}	a_{m2}	a_{mj}	a_{mn}

- Graphing Calculator instructions

1. Press [ALPHA][ZOOM] to create a matrix
 2. Click [x-1] to identify the inverse of matrix A
 3. Enter the constant matrix, B.
 4. Press [ENTER] to evaluate the variable matrix, X.
- **Example 1:** The Alfred Company is a plastic container and wooden container maker. The following plastic containers were developed on a specific day: 900 with a capacity of 30 pounds (lb), 450 with a capacity of 10 pounds, and 200 with a capacity of 20 pounds. On the same day, the following wooden containers were produced: 530 with a capacity of 30 pounds, 670 with a capacity of 10 pounds, and 880 with a capacity of 20 pounds. Find a matrix 2 by 3 that represents this info. To represent the same data, find a 2 by 3 matrix (A)



	30	20	10
Plastic	900	200	450
Wooden	530	880	670

● Operations with matrices

- **Scalar Multiplication**
 - Increase each number in the matrix by the factor
 - Let's call the matrix in the previous function matrix 'A'
 - If the question was asking 'What is 4A?', you would just multiply each number in the matrix by 4.
- **Addition/subtraction**
 - Dimensions of the matrices must match!
 - This means if one matrix is 2x2 you cannot add it to a 3x2 matrix
- **Multiplication**
 - When multiplying, you do rows x columns!
 - Associative and distributive properties apply to matrix multiplication, but NOT commutative property!
- **The Identity Matrix**
 - Equivalent to multiplying by 1, you'll always end up with the original matrix.

- When multiplying by an identity matrix, you'll have an I with a superscript that tells you how many rows and columns there are.

$$C = \begin{bmatrix} 9 & 0 \\ -3 & 6 \end{bmatrix} \quad I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$CI_2 = ?$$

$$\begin{bmatrix} 9 & 0 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ -3 & 6 \end{bmatrix}$$

● Solving Linear Systems Using Inverse Matrices

- The first column in the matrix is for the x value in the polynomial, the second is for the y, and the third is for the x

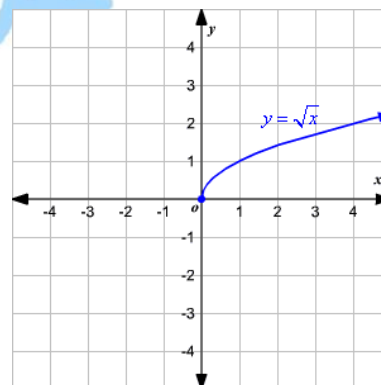
$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Unit 2

Library of Functions

● Square Root

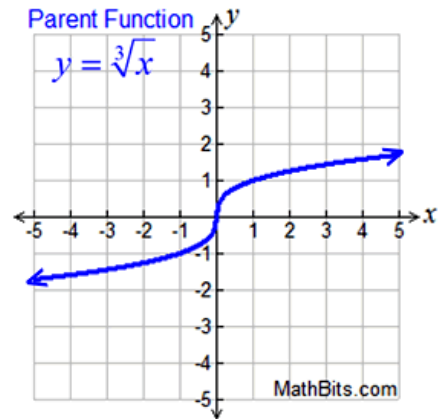
- Parent function: $f(x) = \sqrt{x}$
- 3 common points:
 - (0,0)
 - (1,1)
 - (4,2)
- Domain: $[0, \infty)$



- Range: $[0, \infty)$

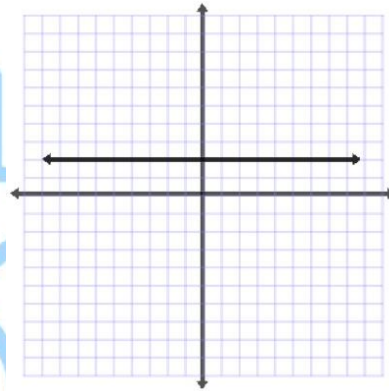
● Cube Root

- Parent function: $f(x) = \sqrt[3]{x}$
- 3 common points:
 - $(-1, 1)$
 - $(0, 0)$
 - $(3, 3)$
- Domain: $(-\infty, \infty)$
- Range: $(-\infty, \infty)$



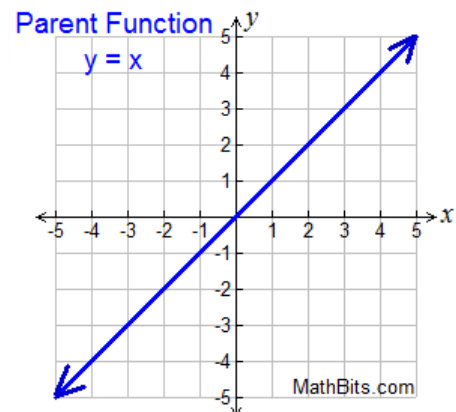
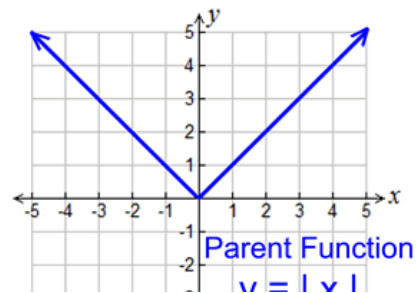
● Constant

- Parent function: $f(x) = b$ where b is a constant
- 3 common points:
 - $(-1, b)$
 - $(0, b)$
 - $(1, b)$
- Domain: $(-\infty, \infty)$
- Range: $\{y | y = b\}$ [b]



● Absolute

- Parent function: $f(x) = |x|$
- 3 Common Points
 - $(-1, 1)$
 - $(0, 0)$
 - $(1, 1)$
- Domain: $(-\infty, \infty)$
- Range: $[0, \infty)$



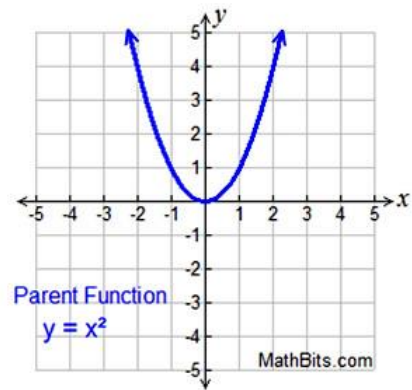
● Linear/Identity

- Parent Function: $f(x) = x$
- 3 Common Points:
 - $(-1, -1)$

- (0,0)
- (1,1)
- Domain: $(-\infty, \infty)$
- Range: $(-\infty, \infty)$

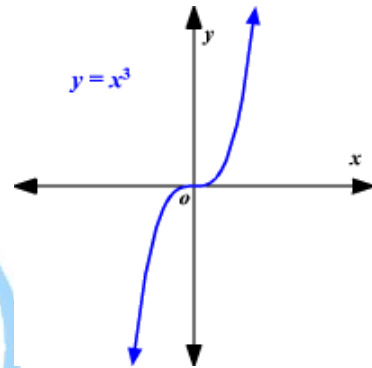
● Square/Quadratic

- Parent Function: $f(x) = x^2$
- 3 Common Points:
 - (-1,1)
 - (0,0)
 - (1,1)
- Domain: $(-\infty, \infty)$
- Range: $[0, \infty)$



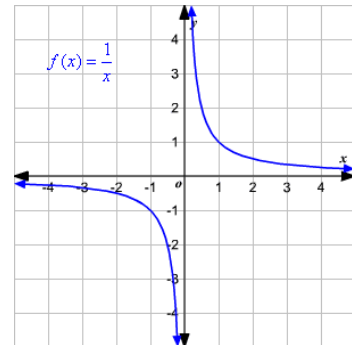
● Cubic

- Parent Function: $f(x) = x^3$
- 3 Common Points:
 - (-1,-1)
 - (0,0)
 - (1,1)
- Domain: $(-\infty, \infty)$
- Range: $(-\infty, \infty)$



● Rational/Reciprocal

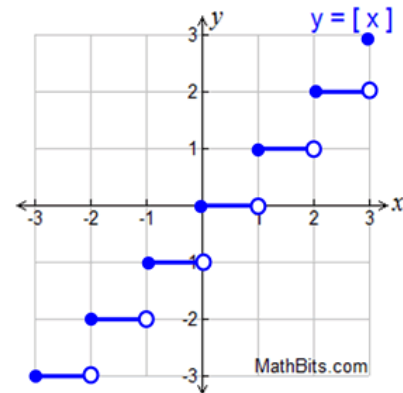
- Parent Function: $f(x) = \frac{1}{x}$
- 5 Common points
 - (-1, 1)
 - (1, 1)
 - (2, 1/2)
 - (-2, 1/2)
 - (1/2, 2)
- Domain: $(-\infty, 0) \cup (0, \infty)$



- Range: $(-\infty, 0) \cup (0, \infty)$

● Greatest Integer

- Parent Function: $[[x]] = \text{int}(x)$
- Common Points:
 - $(0, 1) 0$
 - $(1, 2) 1$
- Domain: $(-\infty, \infty)$
- Range: $\{y \mid y = z\}$



Transformations

- Example Parent function: $f(x) = x^2$
 - $y = a(bx + c)^2 + d$
- Vertical Stretch: $a > 1$
- Vertical Compression: $0 < a < 1$
- Reflection across x-axis: $-a$
- Horizontal Compression: $b > 1$
- Horizontal Stretch: $0 < b < 1$
- Reflection across y-axis: $-b$
- Shifts Left: $+c$
- Shifts Right: $-c$
- Shifts Up: $+d$
- Shifts Down: $-d$

Polynomial Functions

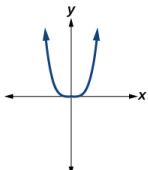
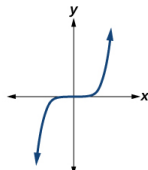
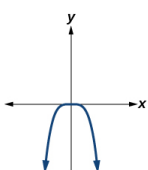
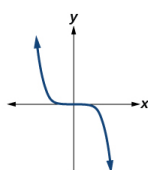
- $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
 - The a 's are the **coefficients**.
 - n 's are **exponents** (all positive integers and 0)
- How to determine if a function is polynomial:

- If there is a +, −, or × sign in the exponent, it is a **polynomial**..
- If there is a **division sign**, then it is a **rational function**.
- It must have a **domain of all real numbers**.
- Let's look at some examples:
 - $f(x) = 2 - 3x^4$
 - This is polynomial because there is a degree of 4.
 - $h(x) = 0$
 - This is a polynomial because there is no degree.
 - $g(x) = \sqrt{x}$
 - You can rewrite this function as $x^{1/2}$
 - There is a fraction in the exponent.
 - $f(x) = \frac{x^2-2}{x^3-1}$
 - The numerator and denominator themselves are polynomials. However, as a whole, it is a rational function, because it **doesn't have a domain of all real numbers**
 - If the x - value was zero it would be undefined.

● Power functions

- A power function is also known as a **monomial**, an expression that is made up of **only one term**.
- $f(x) = ax^n$
- Properties of power functions when n is **even**:

- Symmetry: y-axis
- **Domain:** $(-\infty, \infty)$
- Common Points: $(-1, a)$ $(0, 0)$ $(1, a)$
- Properties of power functions when n is **odd**:
 - Symmetry: Origin
 - **Domain:** $(-\infty, \infty)$
 - Common Points: $(-1, -a)$ $(0, 0)$ $(1, a)$

Even power	Odd power
 <p>$x \rightarrow -\infty, f(x) \rightarrow \infty$ and $x \rightarrow \infty, f(x) \rightarrow \infty$</p>	 <p>$x \rightarrow -\infty, f(x) \rightarrow -\infty$ and $x \rightarrow \infty, f(x) \rightarrow \infty$</p>
 <p>$x \rightarrow -\infty, f(x) \rightarrow -\infty$ and $x \rightarrow \infty, f(x) \rightarrow -\infty$</p>	 <p>$x \rightarrow -\infty, f(x) \rightarrow \infty$ and $x \rightarrow \infty, f(x) \rightarrow -\infty$</p>

● Roots, Zeroes, and Intercepts:

- **Zero of a function:** x -value that make the y -value $= 0$
- **The root of a function:** The solutions of the polynomial equations when the function equals zero.
- **X-intercepts of a function:** These are the points where the graph crosses the x -axis.
- **Let's Practice!**
 - What are the roots of $f(x) = x^2 + x - 6$?
 - Step 1: Factor
 - $f(x) = (x+3)(x-2)$
 - Step 2: set ' $x+3$ ' and ' $x-2$ ' equal to zero and solve for x
 - $x + 3 = 0 \Rightarrow x = -3$
 - $x - 2 = 0 \Rightarrow x = 2$
 - Answer: The roots of the function are -3 and 2
 - Find a polynomial of degree 3 whose zeros are $-3, 5, -7$
 - Degree 3 means that 3 is the term with the largest power
 - Answer: $f(x) = (x+3)(x+7)(x-5)$

● Multiplicity

- This is when the same factor occurs **more than once**.
- Example 1: $f(x) = (x-2)(x+3)^4(x-5)^7$
 - What are the roots? How many times do they occur?
 - The roots are $2, -3$, and 5
 - 2 occurs 1 time

- -3 occurs 4 times
- -5 occurs 7 times
- If you add how many times each of the 3 roots occur, you will find the **degree of the function**.
 - For this function, it is 12

○ **Graphing:**

- When the multiplicity is even, it bounces off the x-axis.
- When the multiplicity is odd, the graph crosses the x-axis.

○ **Example 2: $f(x) = x^2(x-2)$**

- Roots? 2 and 0
- Multiplicity
 - The multiplicity of 2 is 1, so it crosses the x-axis.
 - The multiplicity of 0 is 2, so it bounces off the x-axis.



Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Sample Point	$x = -1$	$x = 1$	$x = 3$
Function Value	$f(-1) = -3$	$f(1) = -1$	$f(3) = 9$
Above or below? (x-axis)	Below, the 'y' value is negative	below	above
Point on graph	$(-1, -3)$	$(1, -1)$	$(3, 9)$

● **Turning Points**

- If f is a degree n polynomial function, then f has **$n-1$ turning points at most**.
- Turning points = degree - 1

● **End Behavior**

- If x is a **really large number**, the graph of the polynomial will look like the graph of the power function.
- Let's practice all these with these new concepts:

- Example 3: $f(x) = x^3 + x^2 - 12x$

■ Y-intercept

- Step 1: substitute 0 in for 'x'
 - $f(0) = 0^3 + 0^2 - 12(0)$
- Answer: (0,0)

■ X-intercepts

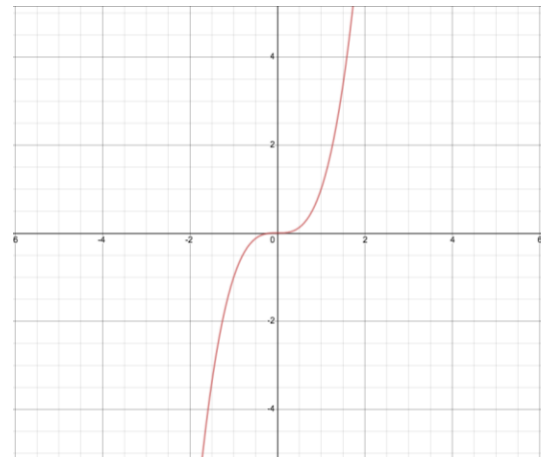
- Step 1: Factor out a what is common
 - All the terms have x in common, so you can bring that out
 - $f(x) = x(x^2 + x - 12)$
- Step 2: Factor what is in the parentheses further
 - $f(x) = x(x + 4)(x - 3)$
- Answers:
 - (0,0) root: 0 multiplicity: 1
 - ((-4,0) root: -4 multiplicity: 1
 - (3,0) root: 3 multiplicity: 1

■ Turning points

- Step 1: determine the degree the of the polynomial
 - Degree = 3
- Step 2: subtract 1 from the degree
 - $3 - 1 = 2$
- Answer: 2

■ End behavior

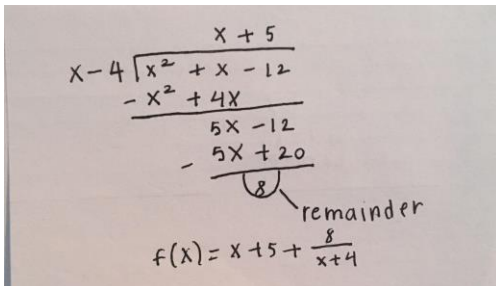
- The power function is $y = x^3$



Real Zeros of a Polynomial Function

- **Theorem:** If the polynomial functions are denoted by $f(x)$ and $g(x)$, and if $g(x)$ is not the zero polynomial, then there are special polynomial functions such that:

- $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$ or $f(x) = q(x)g(x) + r(x)$
- Example 1: $f(x) = \frac{x^2+x-12}{x-4}$



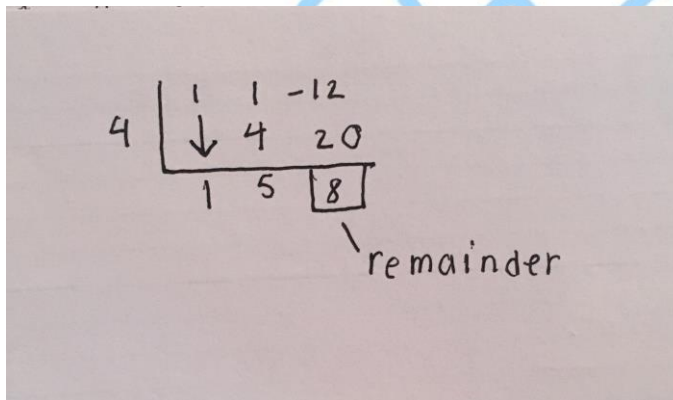
$$\begin{array}{r} x+5 \\ x-4 \overline{) x^2 + x - 12} \\ \underline{-(x^2 + 4x)} \\ 5x - 12 \\ \underline{-(5x + 20)} \\ 8 \end{array}$$

remainder

$$f(x) = x + 5 + \frac{8}{x-4}$$

- **Remainder Theorem:** Let f be a polynomial function. If $f(x)$ is divided by $x-c$, then the remainder is $f(c)$

- Example 2: Find the remainder using the remainder theorem from $F(x) = \frac{x^2+x-12}{x-4}$



$$\begin{array}{r|rrr} 4 & 1 & 1 & -12 \\ & \downarrow & 4 & 20 \\ \hline & 1 & 5 & 8 \end{array}$$

remainder

- **Factor Theorem:** Let f be a polynomial function. Then $x-c$ is a factor of $f(x)$ if and only if $f(c) = 0$

- What does that mean?
 1. If $f(c) = 0$, then $x-c$ is a factor of $f(x)$
 2. If $x-c$ is a factor of $f(x)$, then $f(c) = 0$
- Example 3:

$$\begin{array}{r|rrrr}
 1 & 2 & -1 & 2 & -3 \\
 & \downarrow & & & \\
 & 2 & 1 & 3 & 0
 \end{array}$$

plug 1 in into $x-1$
 $(x-1)$ is a factor
 $x=1$ is a zero
 $f(x) = (x-1)(2x^2 + x + 3)$
 remainder

● Real Zeros

- **Theorem:** A polynomial function may not have more real zeros than the highest degree.
- **Descartes' rule of signs:**
 - Let f denote a polynomial function written in standard form:
 - The number of positive real zeros of f either equals the **number of variations in the sign of the coefficients of $f(x)$** or else equals that number less and even integer.
 - The number of negative real zeros of f either equals the **number of variations in the sign of the nonzero coefficients of $f(-x)$** or else equals that number less an even integer.
- **Example 4:** Discuss the real zeros of $f(x) = 3x^6 - 4x^4 + 3x^3 + 2x^2 - x - 3$
 - Zeros: maximum 6 because that is the degree
 - Positive real numbers/sign changes: 3 or 1
 - Only change signs of odd degree terms
 - Negative Real: 3 or 1
 - Imaginary: 0 or 2 or 4

● Rational Zeros Theorem:

- Let f be a polynomial function of degree 1 or higher of the form
 - $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
- Where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f , then p must be a factor of a_0 (Constant), and q must be a factor of a_n
- **Example 5:** List the potential rational zeros of $f(x) = 2x^3 + 11x^2 - 7x - 6$
 - *Degree tells you number of factors*

- p: 2,3,1,6
- q: 1,2
- $\frac{p}{q} = \pm(1,2,3,6, \frac{1}{2}, \frac{3}{2})$

- 12 numbers because it is + or -

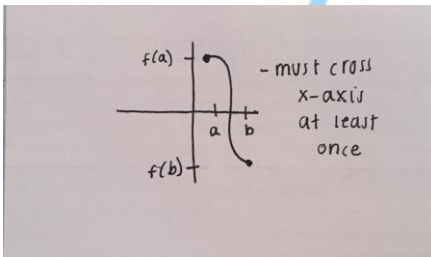
- **Theorem:** Each polynomial function (with real coefficients) can be individually factored into the product of linear factors and/or square factors.

- Linear ex; $x+2$
- Irreducible ex; x^2-2

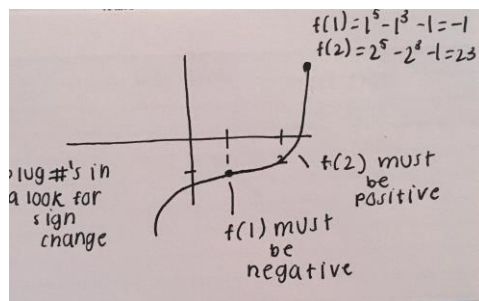
- **Corollary:** A polynomial function (with real coefficients) of odd degree has at **least one real zero**.

- Why?
 - Because it ***crosses the x-axis***
 - It has an ***odd number of zeros***

- **Intermediate Value Theorem:** Let f indicate a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, then there is at least one zero between a and b .



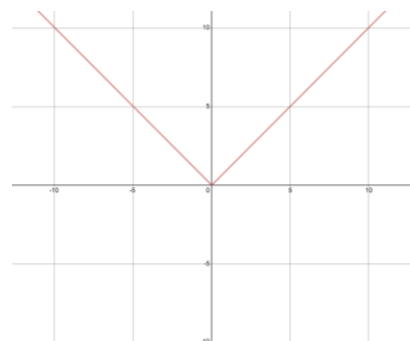
- **Example 6:** Show that $f(x) = x^5 - x^3 - 1$ has a zero between 1 and 2



Piecewise-defined Function

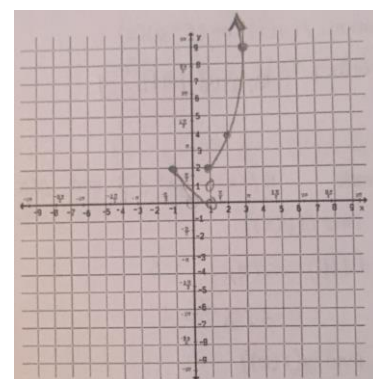
- One of the **most common piecewise** defined functions is....

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Example 1:

- What is $f(0)$?
 - 1
- What is $f(1)$?
 - ± 2
- What is $f(2)$?
 - 4
- What is the domain of $f(x)$?
 - $[-1, \infty)$
- What is the range of $f(x)$?
 - $(0, \infty)$



Composite Functions

- Notation: $f(g(x))$ or $(f \circ g)(x)$
 - f “composed with” g
- Example: $f(x) = 2x^2 - 3$ and $g(x) = 4x$
 - Find $f(g(x))$
 - $f(4x) = 2(4x^2) - 3$
 - $32x^2 - 3$
 -

Inverse Functions

- To find the inverse of a function you **switch the place of the x and y values**.
 - Find the inverse of $\{(-3,9), (-2,4), (-1,1), (0,0), (1,1), (2,4), (3,9)\}$

$$\blacksquare \quad \{(9,-3), (4,-2), (1,-1), (0,0), (1,1), (4,2), (9,3)\}$$

- When the inverse of a function is also a function, then the *original function* is **one-to-one**.
- Verifying two functions are inverses:
 - **Example:** $f(x) = 2x+3$, then $f^{-1}(x) = \frac{1}{2}(x-3)$

■ Part 1

- $f(\frac{1}{2}(x-3)) + 3$
- $x-3+x$
- $x \checkmark$

■ Part 2

- $\frac{1}{2}(2x+3) - 3$
- $x + 3 - 3$
- $x \checkmark$

- The inverse of a function is its reflection across $x=y$.

Rational Functions

- $R(x) = \frac{p(x)}{q(x)}$
 - $p(x)$ = polynomial
 - $q(x)$ = polynomial
 - $R(x)$ = ratio of polynomial functions
- Finding the domain:
 - **Problems** occur in the domain of a rational function when $q(x) = 0$
 - Makes the function **undefined**

$$f(x) = \begin{cases} -x+1 & \text{if } -1 \leq x < 1 \\ 2 & \text{if } x = 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

- **Example:**

$$R(x) = \frac{2x^2 - 4}{x + 5}$$

$$x + 5 \neq 0$$

$$\{x \mid x \neq -5\}$$

- **Vertical Asymptote:** As x approaches some number, c , the y -value **approaches** $-\infty$ **or** ∞ . ($x=c$ is also a vertical asymptote.)
- **Theorem:** If r is a real denominator zero, a rational function in the *lowest terms* would have a vertical asymptote $x=r$.

$$f(x) = \frac{x}{x^2 - 4}$$

vertical asymptote: $x = 2$ or $x = -2$

- **Vocabulary**
 - Holes: Zeros of **both** the numerator and the denominator.
 - **X-intercepts:** Zero of the **numerator only**.
 - **Vertical Asymptotes:** Zero of the **denominator only**.
- **End Behavior of Rational Functions:**
 - The quotient of dividing serves as the end behavior.
- **Horizontal Asymptotes**
 - **Theorem:** If the numerator's degree is *less than* the denominator's degree, then the line $y = 0$ is a horizontal asymptote of its graph.
 - **Example:** $f(x) = \frac{x-12}{4x^2+x+1}$
 - Numerator has a degree of 1 and denominator has a degree of 2.
 - Horizontal Asymptote: $y = 0$
 - It is a proper rational function.

- Horizontal Asymptotes besides $y=0$

- Horizontal asymptotes also occur *when the degree of the numerator and denominator are the same.*

- Example: $f(x) = \frac{6x^2+x+12}{3x^2-5x-2}$

- $6/3 = 2$

- Horizontal Asymptote: $y = 2$

- Oblique/slant asymptotes arise when the *numerator degree is greater than the denominator degree by 1.*

- Example: $f(x) = \frac{3x^4-x^2}{x^3-x^2+1}$

Handwritten long division showing the oblique asymptote for the function $f(x) = \frac{3x^4-x^2}{x^3-x^2+1}$. The division yields a quotient of $3x + 3$ and a remainder of $2x^2 - 3x - 3$. The text "oblique asymptote" is written next to the quotient.

- Graphing Rational Functions Checklist

1. Find the Domain of the function by setting the denominator $\neq 0$
2. Locate the intercepts
 - Y: plug 0 in for x
 - X: set the numerator = 0
3. Test for symmetry- even(y-axis) or odd(origin)
 - Does $R(-x) = R(x)$ or $R(-x) = -R(x)$
4. Find vertical asymptotes and/or holes by finding zeros of both the numerator and denominator
5. Find the end behavior (Horizontal or Oblique asymptotes using long division when necessary)
6. Find Points using intervals around x-intercepts, vertical asymptotes, and holes
7. Graph your points
8. Using properties of these to draw the graph by connecting points and approaching asymptotes

Complex Zeros; Fundamental Theorem of Algebra

- Complex number: $a + bi$

- a -real number
- b- ral number
- $i = \sqrt{-1}$ or $i^2 = -1$

- Complex Numbers include **both real and imaginary numbers**.

- Fundamental Theorem of Algebra: Each complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.

- Theorem: Every complex polynomial function $f(x)$ of degree $n \geq 1$ can be factored into n linear factors of the form $f(x) = a_n(x-r_1)(x-r_2)\dots(x-r_n)$ when $a_n, r_1, r_2, \dots, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n zeros

- $n =$ degree, factors, or zeros

- Conjugate Pairs Theorem: Let $f(x)$ be a polynomial whose **coefficients are real numbers**. If $r = a + bi$ is a zero of $f(x)$, then the complex conjugate $\bar{r} = a - bi$ is also a zero of $f(x)$.

- *Imaginary zeros always come in pairs!*

- Example: If $2 - 3i$ is a zero what must also be a zero?

- $2 + 3i$

- Corollary:

- Polynomial f of **odd degree** with real coefficients has **at least one real zero**.

- **Example:** If a polynomial f of degree 5 whose coefficients are real numbers has the real zeros 1, $5i$, and $1+i$, find the remaining two zeros:

- $1-i$
- $-5i$

Polynomial and Rational Inequalities

1. *Solve the inequality for 0*
 - a. **Imagine $f(x) > 0$**
 - b. If $f(x)$ is a **rational inequality**, then express $f(x)$ as the 1 ratio (fraction).
 2. *Find when $f(x) = 0$*
 - a. Imagine a graph of the function $f(x)$; the inequality means to find when the **graph is above/below the x-axis**, so we start by finding the x-intercepts.
 - b. For a rational inequality, find when the numerator(x-intercepts) and denominator(vertical asymptotes) **equal 0**.
 3. *Create intervals around the zeroes of the function(numerator and denominator)*
 4. *Test each interval to see which agree with the original relationship*
 - a. This is to find when the graph is **above the x-axis ($f(x) > 0$)**
 - b. Or when the graph is **below the x-axis ($f(x) < 0$)**
 5. *Check to see if the ends of the intervals are included (hard or soft bracket)*
- **Example:** $x^2 + 7x \geq 0$

Interval	$(-\infty, -7)$	$(-7, 0)$	$(0, \infty)$
Sample Point	$x = -8$	$x = -1$	$x = 1$
Function Value	$y = 8$	$y = -6$	$y = 8$
Above or Below?	above	below	above

Solution: $(-\infty, -7] \cup [0, \infty)$

Unit 3

Exponential Functions

- Law of Exponents

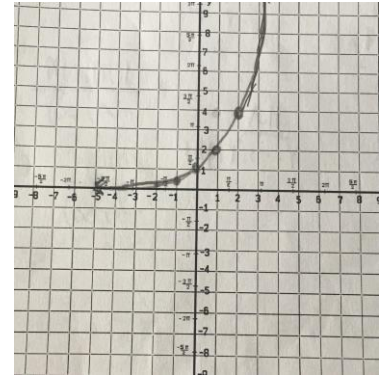
- If m, n, a , and b are real numbers, with $a > 0$ and $b > 0$, then
 - $a^m \times a^n = a^{m+n}$
 - $(ab)^n = a^n \times b^n$
 - $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$
 - $(a^m)^n = a^{mn}$
 - $1^m = 1$
 - $A^0 = 1$
- An exponential function is function of the form: $f(x) = a^x$
 - A is a positive real number when ($a > 0$) and $a \neq 1$
 - a - base
 - x - exponent
 - y - power
- **Theorem:** For an exponential function $f(x) = a^x$, if x is any real number, then

$$\circ \frac{f(x+1)}{f(x)} = \frac{a^{x+1}}{a^x} = a^{(x+1)-(x)} = a^1$$

● Graphing Exponential functions

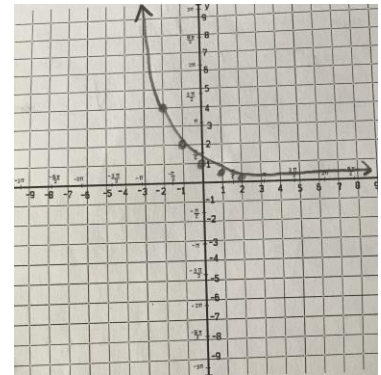
○ Graph $f(x) = 2^x$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- X-intercepts: none
- Y-intercepts: $(0, 1)$
- Horizontal Asymptotes: $y = 0$, as $x \rightarrow -\infty$
- Common Points: $(-1, \frac{1}{2})$ $(0, 1)$ $(1, 2)$
- Continuity: $(-\infty, \infty)$



○ Graph $f(x) = (\frac{1}{2})^x$

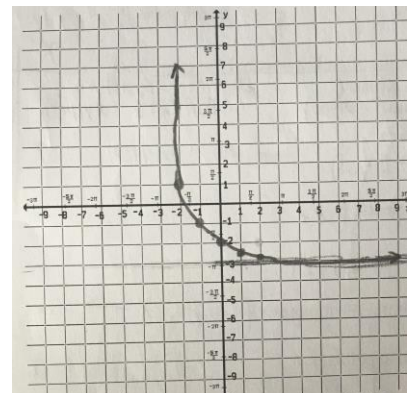
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- X-intercepts: none
- Y-intercepts: $(0, 1)$
- Horizontal Asymptotes: $y = 0$, as $x \rightarrow \infty$
- Common Points: $(-1, 2)$ $(0, 1)$ $(1, \frac{1}{2})$
- Continuity: $(-\infty, \infty)$



● Graphing Using Transformations

○ Graph $y = 2^{-x} - 3$

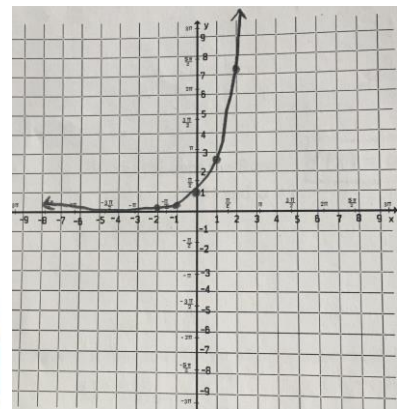
- Parent function: 2^x
- $d = -3$, down 3
- $b = -1$, reflect x's (y-intercept)
- Original points:
 - $(-2, \frac{1}{4})$
 - $(-1, \frac{1}{2})$
 - $(0, 1)$
 - $(1, 2)$



- (2,4)
- New Points:
 - (2, $-2\frac{3}{4}$)
 - (1, $-2\frac{1}{4}$)
 - (0,-2)
 - (-1,-1)
 - (-2,1)
- Horizontal Asymptote: $y = -3$ (right)
- Domain: $(-\infty, \infty)$
- Range: $(-3, -\infty)$

• The base e

- Graph $y = e^x$
 - $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828$
- Points
 - $(-2, 1/e^2)$
 - $(-1, 1/e)$
 - (0,1)
 - (1,e)
 - $(2, e^2)$



• Exponential Equations

- If $a^m = a^n$ then $m = n$
 1. Create like bases
 2. Set exponents equal
- Solve $3^{x+1} = 81$
 - $3^{x+1} = 3^4$
 - $x+1 = 4$
 - $x=3$

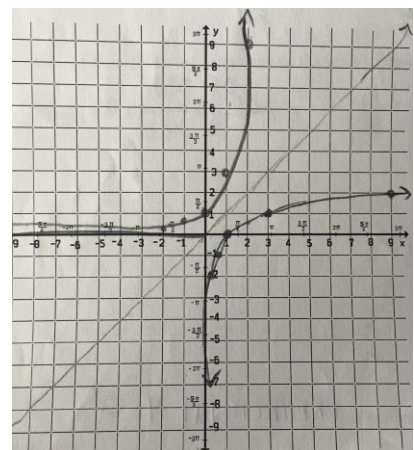
Logarithmic Functions and Properties

- $y = \log_a x$ if and only if $x = a^y$

- exponent = $\log_{\text{base}} \text{power}$
- Exponential
 - $y = a^x$
 - $a = \text{base}$
 - $x = \text{exponent}$
 - $y = \text{power}$
- Example 1: $e^b = 9$
 - $b = \log_e(9)$
- Example 2: $\log_2 16 = 16$
 - $2^x = 16 = 2^4$
 - $x = 4$
- **Domain/Range of a Logarithmic Function**
 - **The inverse function of a logarithmic function is exponential.**
 - Domain of a logarithmic function = Range of Exponential function
 - $(0, \infty)$
 - Range of logarithmic function = Domain of Exponential function
 - $(-\infty, \infty)$
 - Example: Find the domain of $F(x) = \log_2(x - 2)$
 - $x - 2 > 0$
 - $x > 2$
 - $\{x \mid x > 2\}$ or $(2, \infty)$

- **Graphs of Logarithmic Functions**

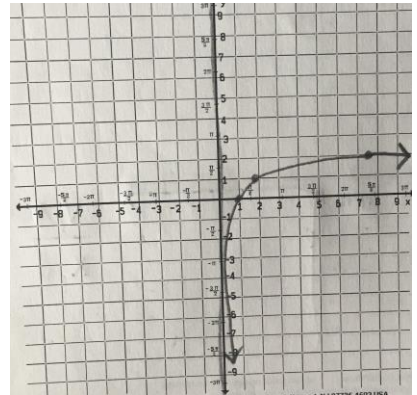
- $f(x) = \log_a x$
- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- X-intercepts: $(1, 0)$
- Y-intercept: None
- Asymptote: Vertical; $x=0$
- Common points: $(1/a, -1)$ $(1, 0)$ $(a, 1)$



- Continuity: $(0, \infty)$

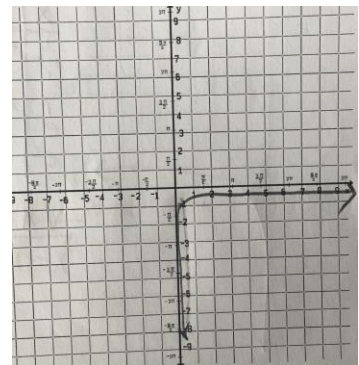
● Natural Logarithms

- $y = \log_e x = \ln x$ if and only if $x = e^y$



● Common Logarithms

- $y = \log x$ if and only if $x = 10^y$



● Solving Logarithmic & Exponential Equations

$$\begin{aligned} \log_3(4x-7) &= 2 & \log_3(9) &= 2 \\ 3^2 &= 4x-7 \\ +7 & & +7 & \\ 4x &= 16 \\ \boxed{x=4} \end{aligned}$$

● Properties of Logarithms

- $\log_a a = 0$
- $\log_a 0 = \text{undefined}$
- $\log_a a = 1$

- $\ln e = 1$
- $\log 10 = 1$
- $a^{\log_a M} = M$
 - Log in exponent.... Same base
 - Cancel out if they have same base
- Log(Product) = sum of logs
 - $\log_a(MN) = \log_a M + \log_a N$
- Log(Quotient) = difference of logs
 - $\log_a \left(\frac{M}{N} \right) = \log_a M - \log_a N$
- $\log_a M^r = r \log_a M$
- Example 1: Write as sum of logs
 - $\log_a(x\sqrt{x^2+1})$
 - $\log_a x + \frac{1}{2} \log_a(x^2+1)$
- If $M = N$, then $\log_a M = \log_a N$
 - And vice versa
- Change-of-Base Formula
 - $\log_a M = \frac{\log_b M}{\log_b a}$
 - b = new base
- Example 2: $\log_5 89$
 - $\frac{\log 89}{\log 5}$

Compound Interest

● Simple Interest Formula: $I = Prt$

- $I \rightarrow$ interest
- $P \rightarrow$ Principle (Money lent/borrowed)
- $r \rightarrow$ Rate (percent)
- $t \rightarrow$ time
 - Annually: 1x year

- Semiannually: 2x year
- Quarterly: 4x year
- Monthly: 12x year
- Daily: 365x year

● Compound Interest Formula

- $A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$
- $P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt}$
 - $A \rightarrow$ Amount (future) P+I
 - $n \rightarrow$ number of times compounded per year

● Continuous Compounding

- If interest was calculated infinitely over and over during the allotted time
 - $A = Pe^{rt}$
 - $P = Ae^{-rt}$

● Effective rate of interest

- **Equivalent annual simple interest rate**, which would yield the same amount after 1 year as compounding
- Needed to compare **different interest rates compounded differently**

Exponential Growth and Decay; Newton's Law; Logistic Models

● Growth and Decay

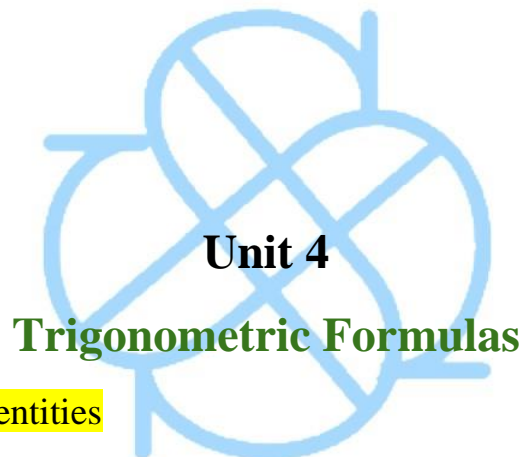
- $A(t) = A_0 e^{kt}$
- $A_0 =$ Initial
- When $k < 0$: **Exponential decay**
- When $k > 0$: **Exponential growth**

● Newton's Law of Cooling

- $u(t) = T + (u_0 - T) \cdot e^{kt}$
 - $u(t)$ - temperature after time
 - u_0 - initial temperature of object
 - T - temperature of environment
 - t - time

● Logistic Growth Model

- $P(t) = \frac{c}{1+ae^{-kt}}$
 - c - carrying capacity



Unit 4

Trigonometric Formulas

● Trigonometric Identities

- $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$
- **Even**
 - $\cos(-\theta) = \cos\theta$
 - $\sec(-\theta) = \sec\theta$
- **Odd**
 - $\sin(-\theta) = -\sin\theta$
 - $\csc(-\theta) = -\csc\theta$

- $\tan(-\theta) = -\tan\theta$
- $\cot(-\theta) = -\cot\theta$
- **Pythagorean**
 - $\cos^2\theta + \sin^2\theta = 1$
 - $\sec^2\theta = 1 + \tan^2\theta$
 - $\csc^2\theta = 1 + \cot^2\theta$

● Sum and Difference Formulas

- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

● Half-Angle Formulas

- $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$
- $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$
- $\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta}$

● Double Angle Formulas

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
- $\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$

Circles and Angles

● Vocab

- **Angles** : 2 rays with a common vertex
 - **Vertex** : **Point** where sides/rays meet, point of rotation
 - **Initial side** : **Beginning** of angle/rotation
 - **Terminal Side** : **End** of angle/rotation
- Positive vs Negative Angles : **Positive angle (counter clockwise)**, **Negative angle (clockwise)**
- Standard position of angles

- Vertex : (0,0)
- Initial Side : Positive X-axis

● Degrees, Minutes, and seconds (D', M', S'')

- DMS to degree Calculator Instructions
 - [2nd][APPS][1] to insert the degree symbol
 - [2nd][APPS][2] to insert the symbol for minutes
 - [ALPHA][+] to insert the symbol for seconds
- Degree to DMS Calculator Instructions
 - Press [MODE], use the arrow keys to highlight DEGREE, and then press [ENTER]
 - Enter the degree measure
 - Press [2nd][APPS][4][ENTER] to convert the degrees to DMS
- 1 degree = 60 minutes and 1 minute = 60 seconds
- Example: Convert $60^{\circ} 25' 32''$ into decimal degrees
 - $60^{\circ} + \frac{25^{\circ}}{60^{\circ}} + \frac{32^{\circ}}{3600} = 60.425^{\circ}$

● Radians

- Radian is a **ratio of 2 distances** as angle grows arc length gets wider
- $\theta = \frac{\text{arc length}}{\text{radius}}$
- 1 radian = $\frac{\text{arc length}}{\text{radius}}$
- Circumference = $2\pi r$
- Full Circle
 - $\theta = \frac{2\pi r}{r} = 2\pi \text{ radians} = 360^{\circ}$
- Half Circle
 - $\pi \text{ radians} = 180^{\circ}$
- Converting radians to degrees
 - $290^{\circ} \cdot \frac{\pi}{180} = \frac{29\pi}{18} \approx 5.06$
- Converting degrees to radians
 - $2.5 \cdot \frac{180^{\circ}}{\pi} = \frac{450}{\pi} = 143.239^{\circ}$
- Arc length

- $s = r\theta$
 - s = arc length
 - r = radius
 - θ = angle in radians

- **Area of sector**

- $A = \frac{1}{2} r^2 \theta$
 - A = area
 - r = radius
 - θ = angle in radians

- **Circular Motion**

- $v = \frac{s}{t}$ (linear speed)
- $\omega = \frac{\theta}{t}$ (angular speed)
- $v = r\omega$ (relationship between linear and angular speed)

Unit Circle and Trigonometric Functions

- **Circle:** $x^2 + y^2 = r^2$
- Unit Circle: $x^2 + y^2 = 1$
 - $r = 1$ in the unit circle
- Center: (0,0)

- **Trigonometric functions:**

- $\sin\theta = \frac{y}{r} = \frac{\text{opp}}{\text{hyp}}$
- $\cos\theta = \frac{x}{r} = \frac{\text{adj}}{\text{hyp}}$
- $\tan\theta = \frac{y}{x} = \frac{\text{opp}}{\text{adj}}$
- $\csc\theta = \frac{r}{y} = \frac{\text{hyp}}{\text{opp}}$
- $\sec\theta = \frac{r}{x} = \frac{\text{hyp}}{\text{adj}}$
- $\cot\theta = \frac{x}{y} = \frac{\text{adj}}{\text{opp}}$
- **Quadrantal Angles:** Angles that lie on an axis
 - Value of the 6 trig functions at $\theta = 0$ radians

- $\sin\theta = \frac{y}{r} = 0$
- $\cos\theta = \frac{x}{r} = 1$
- $\tan\theta = \frac{y}{x} = 0$
- $\csc\theta = \frac{r}{y} = \text{undefined}$
- $\sec\theta = \frac{r}{x} = 1$
- $\cot\theta = \frac{x}{y} = \text{undefined}$

- **Reference Angle:** an angle that is *directly between* $(0,90)$ or $(0, \frac{\pi}{2})$
- **Periodic Function:** Function that **repeats outputs after a period of time**

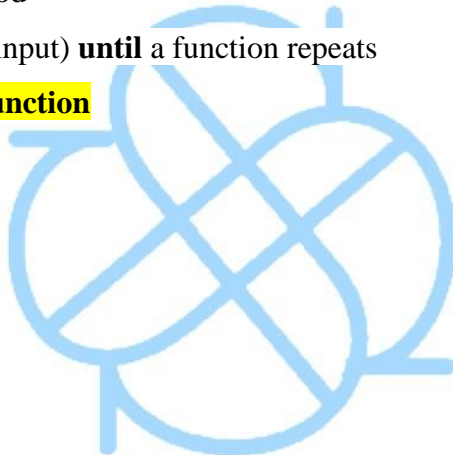
- $f(\theta + p) = f(\theta)$

- $p = \text{period}$

- **Period:** length of time(input) **until** a function repeats

- **Periods of each trig function**

- Sine: 2π
- Cosine: 2π
- Tangent: π
- Cosecant: 2π
- Secant: 2π
- Cotangent: π



- **Quadrants trig identities are positive in**

- I: All
- II: sin and csc
- III: tan and cot
- IV: cos and sec

- **Pneumonic Device:** All Students Take Calculus

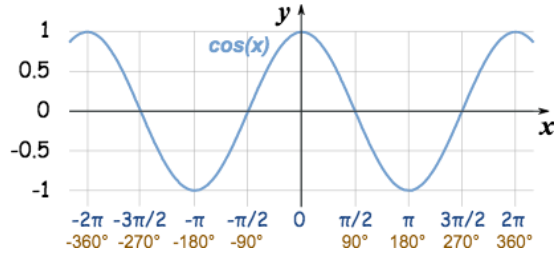
- **How to find the values for the other 5 functions, when given 1 function:**

1. *Identify the quadrant and determine the sign of x, y , and r*
2. *Assign values to 2 of the following x, y , and r*
3. *Use $x^2 + y^2 = r^2$ to find the remaining piece and remember the sign*
4. *Use definitions and identities to find the remaining 4 functions*

Graphing Trig Functions

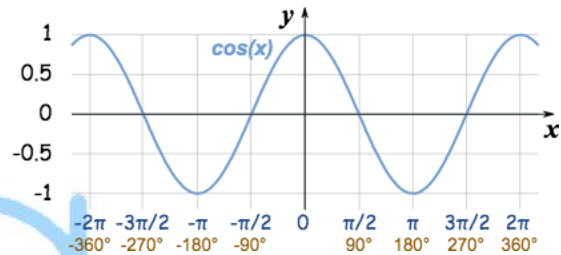
- $y = \sin x$

- Amplitude(A) = 1
- Period(T) = 2π
- Domain: $(-\infty, \infty)$
- Range: $[-1, 1]$



- $y = \cos x$

- Amplitude(A) = 1
- Period(T) = 2π
- Domain: $(-\infty, \infty)$
- Range: $[-1, 1]$

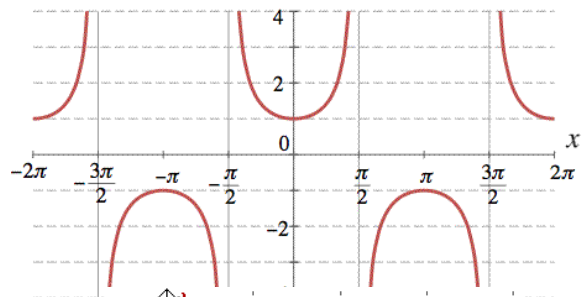


- Transformations

- $Asin(Bx + C) + D$ or $y = Asin(B(x + \frac{C}{B})) + D$
 - Amplitude: $|A|$
 - Period: $\frac{2\pi}{B}$
 - C: Horizontal Shift
 - D: Vertical Shift
 - Interval Width: Period x $\frac{1}{4}$

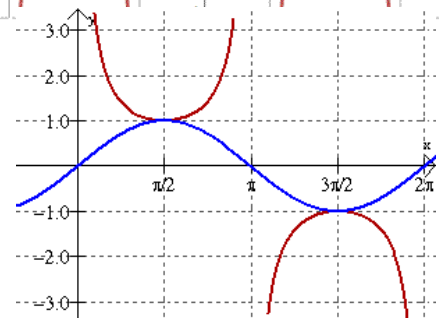
- $y = \sec x$

- Amplitude(A) = A
- Period(T) = $\frac{2\pi}{B}$
- Domain: $\{x \mid x \neq \text{odd multiples of } \frac{\pi}{2}\}$
- Range: $(-\infty, -1] \cup [1, \infty)$



- $y = \csc x$

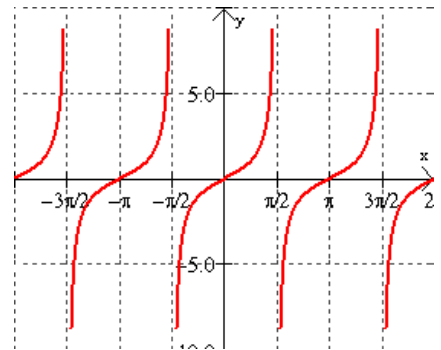
- Amplitude(A) = A
- Period(T) = $\frac{2\pi}{B}$



- Domain: $\{x \mid x \neq \text{integer multiples of } \pi\}$
- Range: $(-\infty, -1] \cup [1, \infty)$

- $y = \tan x$

- Amplitude(A) = A
- Period(T) = $\frac{\pi}{B}$
- Domain: $\{x \mid x \neq \text{odd multiples of } \frac{\pi}{2}\}$
- Range: $(-\infty, \infty)$



- Phase Shifts

- $Bx + C = 0$
 - X = phase shift

Inverse Trig

- $y = \sin^{-1} x$

- Domain: $[-1, 1]$
- Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

- $y = \cos^{-1} x$

- Domain: $[-1, 1]$
- Range: $[0, \pi]$

- $y = \tan^{-1} x$

- Domain: $(-\infty, \infty)$
- Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

- $y = \csc^{-1} x$

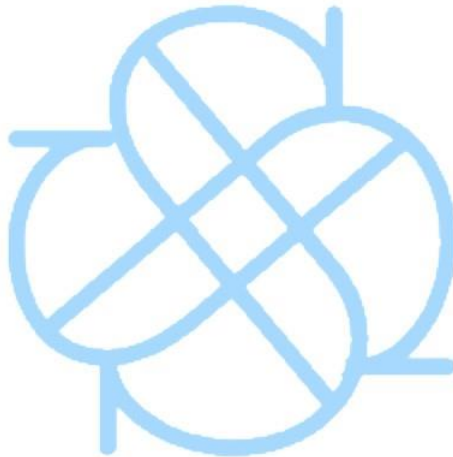
- Domain: $(-\infty, -1] \cup [1, \infty)$
- Range: $[0, \frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$

- $y = \cot^{-1} x$

- Domain: $[-\infty, \infty]$
- Range: $(0, \pi)$

Solving Trigonometric Equations

- $\theta = \frac{\pi}{3} + 2\pi k$ or $\theta = \frac{5\pi}{3} + 2\pi k$
- Isolate trig function
- Example: $2\sin\theta + \sqrt{3} = 0$
 - $\frac{2\sin\theta}{2} = \frac{-\sqrt{3}}{2}$
 - $\sin\theta = \frac{-\sqrt{3}}{2}$
 - Answers:
 - $\theta = \frac{4\pi}{3} + 2\pi k$ or $\theta = \frac{5\pi}{3} + 2\pi k$



Unit 5

Establishing Trigonometric Identities

**** These are very crucial for not only this unit, but also in calculus, so make sure you know them very well**

○ Trig Identities

- $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta}$
 - $\cos\theta = \frac{1}{\sec\theta}$
 - $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$
 - $\sec\theta = \frac{1}{\sin\theta}$
 - $\csc\theta = \frac{1}{\sin\theta}$
 - $\sin\theta = \frac{1}{\csc\theta}$
 - $\sin^2\theta = 1 - \cos^2\theta$
 - $\cos^2\theta = 1 - \sin^2\theta$
 - $\sin^2\theta + \cos^2\theta = 1$
 - $\tan^2\theta + 1 = \sec^2\theta$
 - $1 + \cot^2\theta = \csc^2\theta$
 - $\sin(-\theta) = -\sin(\theta)$
 - $\cos(-\theta) = \cos(\theta)$
 - $\tan(-\theta) = -\tan\theta$
 - $\csc(-\theta) = -\csc(\theta)$
 - $\sec(-\theta) = \sec(\theta)$
 - $\cot(-\theta) = -\cot(\theta)$
- **Establishing Identity Rules**
- Work with only 1 side
 - Choose the more complicated side (usually)
 - Use simplifying techniques and identities to show both sides are congruent
- **Hints for beginning to establish identities**
- Make the form similar for both sides (1 term or 2 terms or 1 fraction...etc)
 - Turn your functions into *sine and cosine*
 - Separate fractions with *single denominator* into multiple fractions
 - Find a common denominator to write as 1 fraction

- Multiply or divide each term in a fraction by the same trig function

○ Examples:

$$\begin{array}{l} \sin^2(-\theta) + \cos^2(-\theta) \stackrel{?}{=} 1 \\ (\sin(-\theta))^2 + (\cos(-\theta))^2 \\ (-\sin\theta)^2 + (\cos\theta)^2 \\ \sin^2\theta + \cos^2\theta \\ \underbrace{\hspace{1.5cm}} \\ 1 = 1 \checkmark \end{array} \quad \begin{array}{l} \downarrow \\ \text{QED} \end{array}$$

$$\begin{array}{l} \tan\theta + \frac{\cos\theta}{1+\sin\theta} = \sec\theta \\ (1+\sin\theta) \cdot \frac{\sin\theta}{(1+\sin\theta) \cdot \cos\theta} + \frac{\cos\theta \cdot (\cos\theta)}{1+\sin\theta \cdot (\cos\theta)} \\ \frac{\sin\theta + \sin^2\theta + \cos^2\theta}{(1+\sin\theta)(\cos\theta)} \\ \frac{\sin\theta + 1}{(1+\sin\theta)(\cos\theta)} = \frac{1}{\cos\theta} = \sec\theta = \sec\theta \\ \text{QED} \end{array}$$

Sum and Difference

- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
- $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

○ Example 1:

$$\cos 75^\circ = \cos(30^\circ + 45^\circ) = \cos 30^\circ \cdot \cos 45^\circ - \sin 30^\circ \cdot \sin 45^\circ$$

$$\blacksquare = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

$$\blacksquare = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$\blacksquare \text{ Answer: } \frac{\sqrt{6} + \sqrt{2}}{4}$$

● Example 2: $\sin \frac{7\pi}{12}$

○ $\sin(\frac{4\pi}{12} + \frac{3\pi}{12})$

○ $\sin \frac{\pi}{3} \cdot \cos \frac{\pi}{4} + \cos \frac{\pi}{3} \cdot \sin \frac{\pi}{4}$

○ $\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

○ $\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

○ Answer: $\frac{\sqrt{6} + \sqrt{2}}{4}$

● Half-Angle Formulas

○ $\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$

○ $\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$

○ $\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}$

● Example: If $\sin \theta = \frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$

○ $\sin(2\theta) = 2\sin\theta\cos\theta$

○ $2 \cdot \frac{3}{5} \cdot \frac{-4}{5} = \frac{-24}{25}$

● Double Angle Formulas

○ $\sin(2\theta) = 2\sin\theta\cos\theta$

○ $\cos(2\theta) = 2\sin\theta\cos\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2$

○ $\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$

● Example: $\cos 15^\circ$

○ $\frac{\sqrt{2 + \sqrt{3}}}{2}$

Law of Sines

- $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

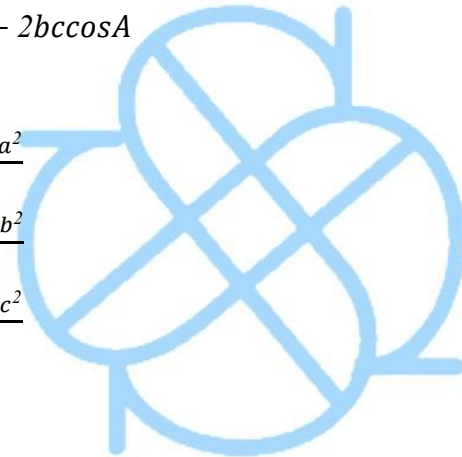
Law of Cosines

- Finding Sides

- $c^2 = a^2 + b^2 - 2ab\cos C$
- $b^2 = a^2 + c^2 - 2ac\cos B$
- $a^2 = b^2 + c^2 - 2bc\cos A$

- Finding Angles

- $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
- $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$
- $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$



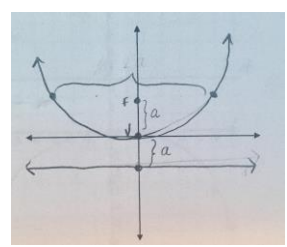
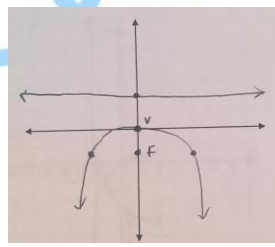
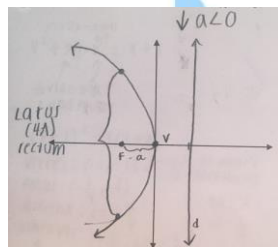
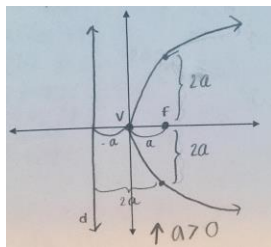
Unit 6

Conic Sections

☆ The graphs are abbreviated 'f' is focus, 'v' is vertex, and 'd' is directrix

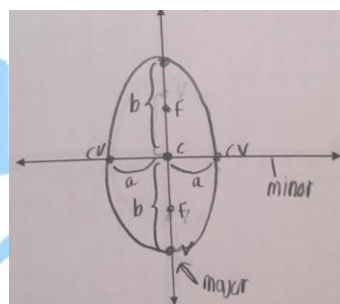
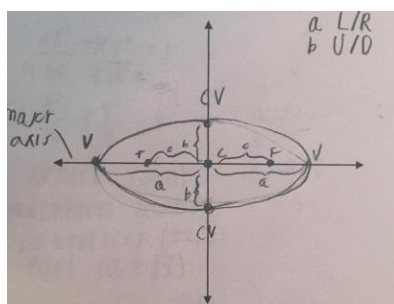
Parabolas are a set of points that are the same distance to a point (focus) and a line (directrix).

Model	$(x - h)^2 = 4a(y - k)$	$(y - k)^2 = 4a(x - h)$
Vertex	(h, k)	(h, k)
Focus	$(h, k + a)$	$(h + a, k)$
Directrix	$y = k - a$	$x = h - a$
Latus Rectum (Points)	$(h \pm 2a, k + a)$	$(h + a, k \pm 2a)$



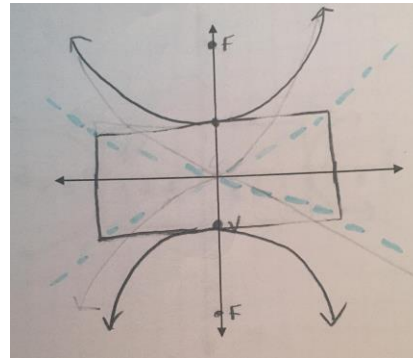
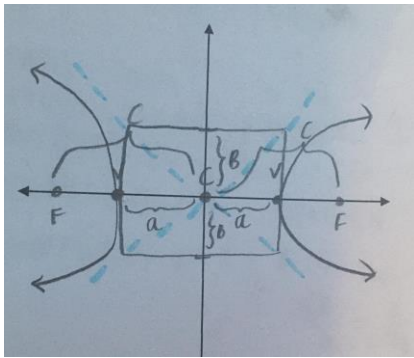
Ellipses are a set of points whose total or sum distance from 2 points (foci) is the same.

Model	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad a > b$	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad a < b$
Center	(h, k)	(h, k)
Vertices	$(h \pm a, k)$	$(h, k \pm b)$
Covertices	$(h, k \pm b)$	$(h \pm a, k)$
Latus Rectum (Points)	$(h \pm c, k)$	$(h, k \pm c)$



A Hyperbola is a set of points whose difference in distance from the 2 foci is the same.

Model	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
Center	(h, k)	(h, k)
Vertices	$(h \pm a, k)$	$(h, k \pm b)$
Foci $c^2 = a^2 + b^2$	$(h \pm c, k)$	$(h, k \pm c)$
asymptotes	$y - k = \pm \frac{b}{a}(x - h)$	$y - k = \pm \frac{b}{a}(x - h)$



Unit 7

Vectors

- **Vectors** have both **direction and magnitude**.
 - The **length** of the ray indicates the **magnitude**.
 - The **arrow** on the ray indicates the **direction**.
- Vectors can be added:
 - **Vector addition** is **commutative and distributive**.
- Vectors can be **multiplied** by a **scalar(dilation)**.
- Vectors can be defined by an **initial and terminal point on a coordinate plane**.
 - $v = ai + bj$
- **Magnitudes of Vectors**
 - $|v| = \text{magnitude \#}$
 - Always nonnegative: $|v| = |v|, |v| > 0$
- **Unit Vector**
 - $u = \frac{v}{|v|} = \frac{\text{vector components}}{\text{vector magnitude}}$
- **Vectors are written with respect to angle measure**.
 - $v = |v|(\cos \theta i + \sin \theta j)$
- **Static Equilibrium:** forces that can be represented by vectors.
 - If the object is at rest, and the sum of all forces acting on the object is zero, then, an object is in static equilibrium.

The Dot Product

- If $v = a_1i + b_1j$ and $w = a_2i + b_2j$, then...
 - $v \cdot w = a_1a_2 + b_1b_2$

- For dot product, the commutative and associative properties can be applied.

- **Angles between vectors:**

- If u and v are nonzero vectors, the angle θ , $0 \leq \theta \leq \pi$, between the two vectors is determined by:

$$\blacksquare \cos\theta = \frac{u \cdot v}{||u|| ||v||}$$

- **Orthogonal/Perpendicular vectors** occur when the angle between two nonzero vectors is $\frac{\pi}{2}$... i.e. $v \cdot w = 0$

- **Applications: work done by a constant force.**

- **Work** = (magnitude of Force vector in direction of movement)(distance)
- **Units:** foot-pounds or newton-meters(Joules)

Parametric Equations

- Let $x = f(t)$ and $y = g(t)$, where f and g are two functions whose **common domain is some interval I** . The collection of points defined by $(x, y) = (f(t), g(t))$ is called a **parametric point**.
- The **equations** $x = f(t)$ and $y = g(t)$ are known as parametric equations.
- **t** is called the **parameter(time)**.

- **Projectile motion**

- The location of an object as it ***travels through space*** depends on:

- **Horizontal motion: Velocity, angle, time**

- $x = v_0 t \cos\theta$

- v_0 = initial velocity

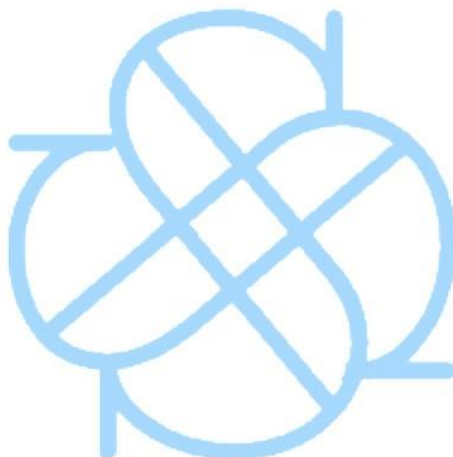
- **Vertical motion: velocity, angle, gravity, time**

- $y = -\frac{1}{2}gt^2 + v_0 t \sin\theta + h$

- h = initial height

- g = gravity = 9.8 m/s^2

Polar Graphs



Polar Graphs Summary

<p align="center">Circle</p> <p align="center">$r = a$</p> <p align="center">Center at the origin</p> <p align="center">Radius=a</p>		<p align="center">Circle</p> <p align="center">Radius: $\frac{a}{2}$</p>		<p align="center">Limaçon</p> <p align="center">$a > b$: no loop</p> <p align="center">Distance from indentation to origin: $a - b$</p>	
		<p align="center">$r = a \sin \theta$</p> <p align="center">Center $\left(\frac{a}{2}, \frac{\pi}{2}\right)$</p> <p align="center">Symmetry: y-axis</p>	<p align="center">$r = a \cos \theta$</p> <p align="center">Center $\left(\frac{a}{2}, 0\right)$</p> <p align="center">Symmetry: x-axis</p>	<p align="center">$r = a \pm b \sin \theta$</p> <p align="center">Symmetry: y-axis</p> <p align="center">x-intercepts: $(a, 0); (a, \pi)$</p> <p align="center">y-intercepts: $\left(a + b, \frac{\pi}{2}\right); \left(a - b, \frac{3\pi}{2}\right)$</p>	<p align="center">$r = a \pm b \cos \theta$</p> <p align="center">Symmetry: x-axis</p> <p align="center">x-intercepts: $(a + b, 0); (a - b, \pi)$</p> <p align="center">y-intercepts: $\left(a, \frac{\pi}{2}\right); \left(a, \frac{3\pi}{2}\right)$</p>
<p align="center">Line through origin</p> <p align="center">$\theta = a$</p>		<p align="center">Cardioid</p>		<p align="center">Limaçon</p> <p align="center">$a < b$: inner loop</p> <p align="center">Length of loop: $b - a$</p>	
<p align="center">Horizontal Line</p> <p align="center">$r \sin \theta = a$</p> <p align="center">Same as: $y = a$</p>	<p align="center">Vertical Line</p> <p align="center">$r \cos \theta = a$</p> <p align="center">Same as: $x = a$</p>	<p align="center">$r = a \pm a \sin \theta$</p> <p align="center">Symmetry: y-axis</p> <p align="center">x-intercepts: $(a, 0); (a, \pi)$</p> <p align="center">y-intercepts: $\left(2, \frac{\pi}{2}\right); \left(2, \frac{3\pi}{2}\right)$</p>	<p align="center">$r = a \pm a \cos \theta$</p> <p align="center">Symmetry: x-axis</p> <p align="center">x-intercepts: $(2a, 0); (0, \pi)$</p> <p align="center">y-intercepts: $\left(2, \frac{\pi}{2}\right); \left(2, \frac{3\pi}{2}\right)$</p>	<p align="center">$r = a \pm b \sin \theta$</p> <p align="center">Symmetry: y-axis</p> <p align="center">x-intercepts: $(a, 0); (a, \pi)$</p> <p align="center">y-intercepts: $\left(a + b, \frac{\pi}{2}\right); \left(a - b, \frac{3\pi}{2}\right)$</p>	<p align="center">$r = a \pm b \cos \theta$</p> <p align="center">Symmetry: x-axis</p> <p align="center">x-intercepts: $(a + b, 0); (a - b, \pi)$</p> <p align="center">y-intercepts: $\left(a, \frac{\pi}{2}\right); \left(a, \frac{3\pi}{2}\right)$</p>
<p align="center">Rose</p> <p align="center">b is odd: b number of petals</p> <p align="center">b is even: $2b$ number of petals</p> <p align="center">Petal Length: a</p>		<p align="center">Lemniscate</p> <p align="center">Length of propeller: a</p>			
<p align="center">$r = a \sin(b\theta)$</p> <p align="center">Symmetry: y-axis</p> <p align="center">To Graph: Plot $\left(a, \frac{\pi}{2}\right)$ or $\left(a, \frac{3\pi}{2}\right)$ and evenly distribute other petals</p>	<p align="center">$r = a \cos(b\theta)$</p> <p align="center">Symmetry: x-axis</p> <p align="center">To Graph: Plot $(a, 0)$ and evenly distribute other petals</p>	<p align="center">$r^2 = a^2 \sin(2\theta)$</p> <p align="center">Symmetric: $\theta = \frac{\pi}{4}$</p> <p align="center">To Graph: Plot $\left(a, \frac{\pi}{4}\right); \left(a, \frac{5\pi}{4}\right)$</p>	<p align="center">$r^2 = a^2 \cos(2\theta)$</p> <p align="center">Symmetric: x-axis</p> <p align="center">To Graph: Plot $(a, 0); (a, \pi)$</p>		

Unit 8

Arithmetic Sequences/Series and Geometric Sequences/Series

- A **sequence** is a function whose **domain is the set of positive functions**.

- The **terms for the sequence** $f(n) = \frac{1}{n}$

- n = input, position
- f_n = term

• Factorials

- $n! = n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1$ where $0! = 1$ and $1! = 1$

• Recursive Formulas

- A sequence that uses previous terms

• Properties of Sequences

- $\sum_{k=1}^n c = c + c + c + \dots + c = cn$
- $\sum_{k=1}^n (ca_k) = ca_1 + ca_2 + ca_3 + \dots + ca_n = c(a_1 + a_2 + \dots + a_n) = c \sum_{k=1}^n a_k$
- $\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n (a_k) + \sum_{k=1}^n (b_k)$
- $\sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n (a_k) - \sum_{k=1}^n (b_k)$
- $\sum_{k=1}^n (a_k) = \sum_{k=1}^j (a_k) + \sum_{k=j+1}^n (a_k); 0 < j < n$
- $\sum_{k=1}^n k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$
- $\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

• Example:

- $\sum_{k=1}^9 (3k) = 3 \sum_{k=1}^9 k = 3 \cdot \left(\frac{9(9+1)}{2}\right) = 135$

- Arithmetic Sequences: differences between successive terms is always the same number

- $a_1 = a$
- $a_n = a_{n-1} + d$

■ d : common difference(slope)

■ a : first term

- **Example:** 4, 7, 10, 13, ...

- $7 - 4 = 3$
- $10 - 7 = 3$
- $13 - 10 = 3$
- **Answer:**

■ $d = 3$ & $a = 4$

- **Arithmetic Sequence Theorem:** $a_n = a + (n - 1)d$

- **Example:** find the 13th term of the sequence: 2, 6, 10, 14, 18, ...

- $d = 4, a = 2$
- $a_{13} = 2 + (13 - 1)4 = 50$

- **Arithmetic Series Theorem:** Let $\{a_n\}$ be an arithmetic sequence with first term a and a common difference d . The sum S_n of the first n terms of $\{a_n\}$ is

- $S_n = \frac{n}{2} [2a + (n - 1)d] = \frac{n}{2} (a + a_n)$

- Find the sum S_n of the first n terms of the following sequence (in terms of 'n')

- $\{3n + 5\}$
- $S_n = \frac{n}{2} [8 + 3n + 5] = \frac{n}{2} (3n + 13) = \frac{3n^2 + 13n}{2}$
- **Answer:** $\frac{3n^2 + 13n}{2}$

- **Geometric Sequence:** ratio of successive terms is the **same nonzero number**

- $a_1 = a$

■ a = 1st term

- $a_n = ra_{n-1}$

■ r = common ratio

- **Geometric Sequence Theorem:** For a geometric sequence $\{a_n\}$ whose first term is a and whose common ratio is r , the n -th term is determined by

- $a_n = ar^{n-1}$

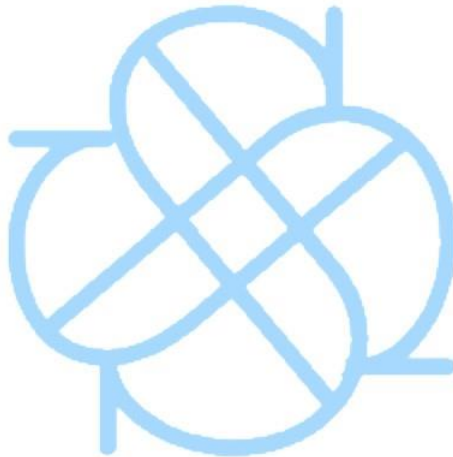
- Adding the first n terms of a geometric sequence

- $S_n = a \frac{1 - r^n}{1 - r}$

- **Infinite Geometric Series Theorem**

- The sum of an infinite geometric series is:

- $\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$



Let's Begin Calculus Now!!

Unit 9

Introduction to the Limit Process and a Preview of Calculus

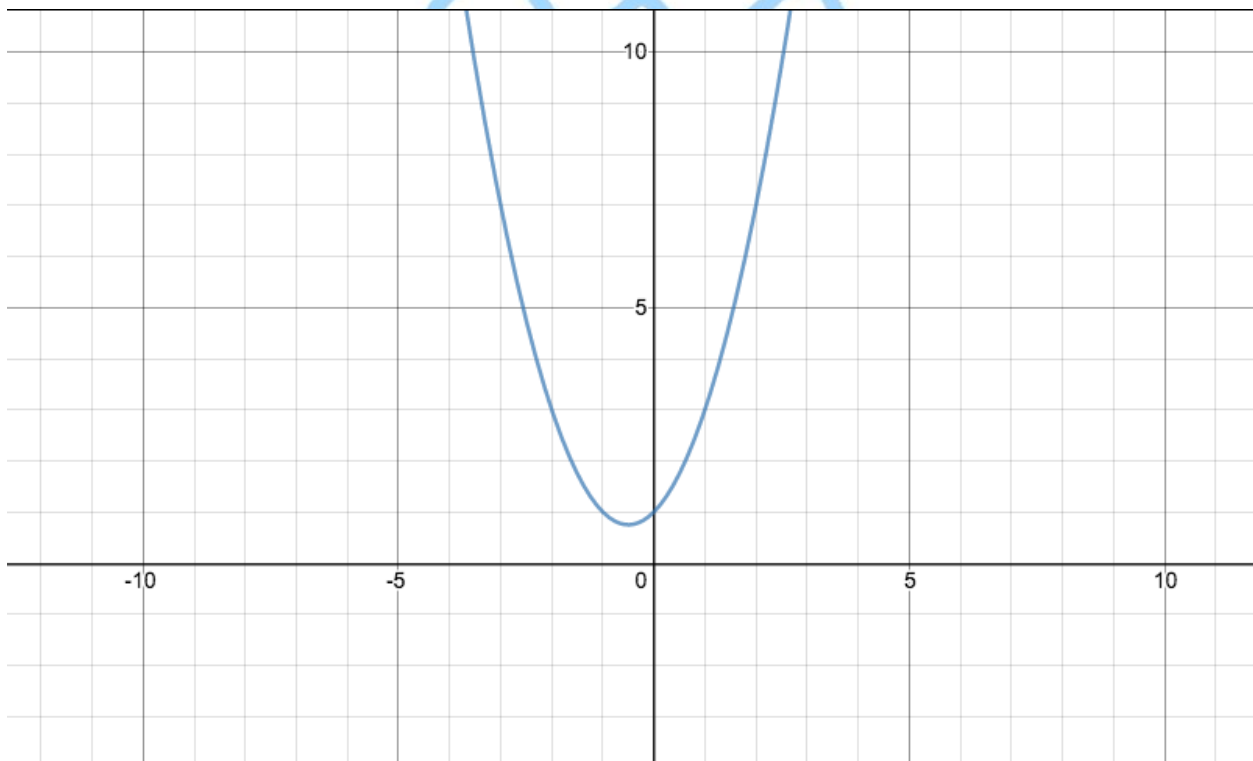
- Calculate the **slope of the secant line** using average rate of change from point (1,2) to (3,10)

- Use the slope formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$
- $m = \frac{10 - 2}{3 - 1} = \frac{8}{2} = 4$
- You can use the slope of secant line to approximate the slope of the tangent line

Finding Limits Graphically and Numerically

- **Limits** are paths that the graph is **approaching**.
- Example: consider $f(x) = \frac{x^3 - 1}{x - 1}$
 - If you substitute $x = 1$ into the function its domain would be undefined

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
f(x)	2.3125	2.71	2.9701	2.997001	3?	3.003001	3.0301	3.31	3.8125



● Limit Notation:

- $\lim_{x \rightarrow c} f(x) = L$

- As x approaches “ c ”, $f(x)$ approaches “ L ”
- The limit of $f(x)$ is L as x approaches point (or hole) from both directions
- **Formal Definition of a Limit:** Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement:
 - $\lim_{x \rightarrow c} f(x) = L$
 - Means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if
 - $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$
 - $\varepsilon = \text{epsilon}$
 - $\delta = \text{delta}$

Evaluating Limits

Analytically

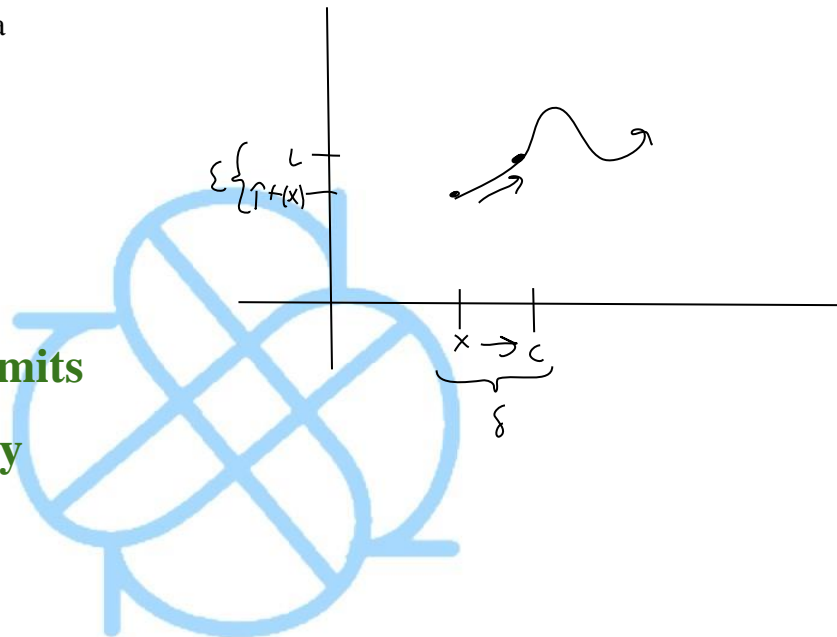
- $\lim_{x \rightarrow c} b = b$
- $\lim_{x \rightarrow c} x = c$
- $\lim_{x \rightarrow c} x^n = c^n$
- **Theorem:** let b and c be real numbers, let n be a positive integer and let f and g be functions with following limits.

$$\lim_{x \rightarrow c} f(x) = L$$

and

$$\lim_{x \rightarrow c} g(x) = K$$

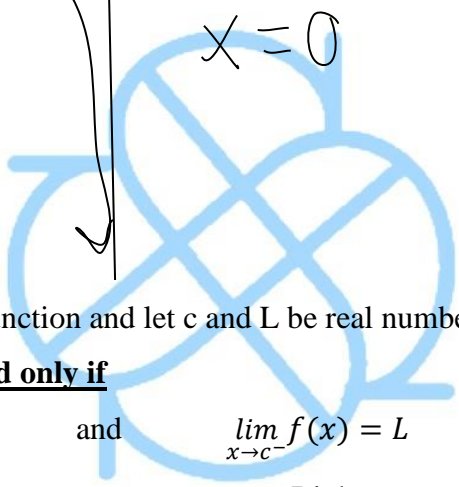
- **Scalar Multiplication:** $\lim_{x \rightarrow c} [bf(x)] = bL$
- **Sum or Difference:** $\lim_{x \rightarrow c} [f(x) + g(x)] = L \pm K$
- **Product:** $\lim_{x \rightarrow c} [f(x)g(x)] = LK$
- **Quotient:** $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{K} \quad K \neq 0$
- **Power:** $\lim_{x \rightarrow c} [f(x)]^n = L^n$



- Example: $\lim_{x \rightarrow 2} (4x^2 + 3) =$
 - $\lim_{x \rightarrow 2} 4x^2 + \lim_{x \rightarrow 2} 3$
 - $4(\lim_{x \rightarrow 2} 4)^2 + \lim_{x \rightarrow 2} 3$
 - $4(2)^2 + 3$
 - $16 + 3$
 - Answer: 19
- **Direct Substitution Theorem:** If p is a polynomial function and c is real number, then
 - $\lim_{x \rightarrow c} p(x) = p(c)$
- Example: $\lim_{x \rightarrow 0} \tan x$
 - $\tan(0) = 0$

Continuity and One-Sided Limits

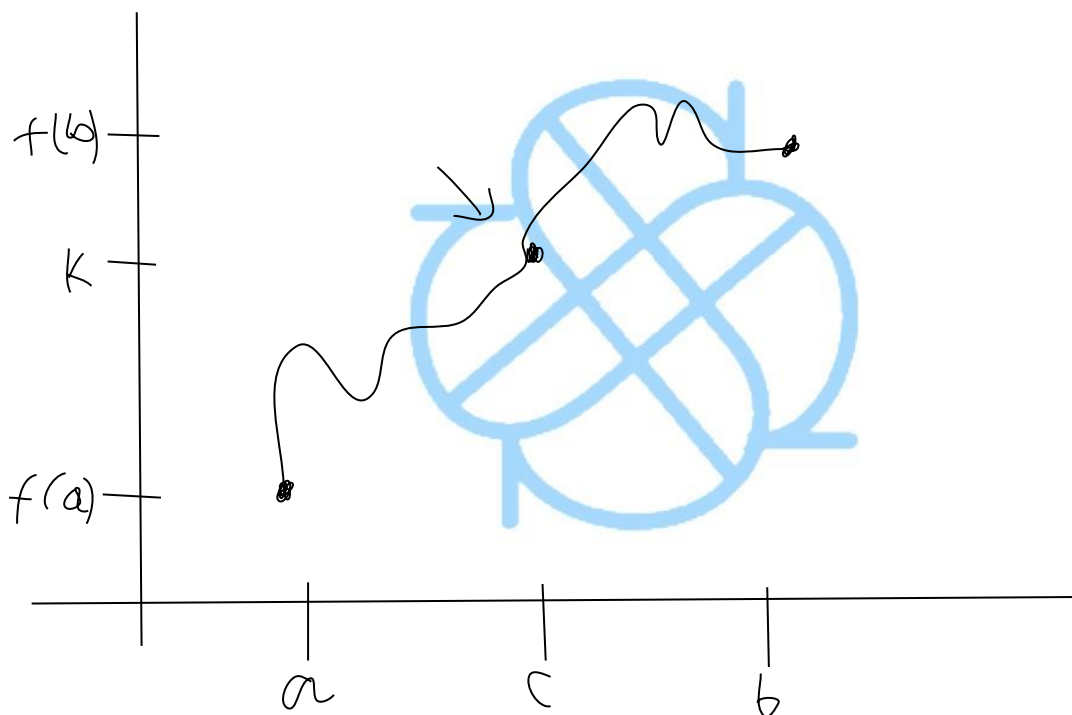
- A function f is continuous at c if the following 3 conditions are met:
 - **$f(c)$ is defined**
 - **$\lim_{x \rightarrow c} f(x)$ exists**
 - **$\lim_{x \rightarrow c} f(x) = f(c)$**
- Example: $f(x) = \frac{1}{x}$



- Theorem:** Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L **if and only if**
 - $\lim_{x \rightarrow c^-} f(x) = L$ and $\lim_{x \rightarrow c^+} f(x) = L$

Left
Right
- A function is **continuous** on the **closed interval $[a,b]$** if it is **continuous** on the open interval (a,b) and
 - $\lim_{x \rightarrow a^-} f(x) = f(a)$ and $\lim_{x \rightarrow b^+} f(x) = f(b)$
- Theorem:** If **b is a real number and f and g are continuous at $x = c$** , then the following functions are also continuous at c
 - Scalar Multiple:** bf
 - Sum and Difference:** $f \pm g$
 - Product:** fg
 - Quotient:** $\frac{f}{g}$

- **Theorem:** If g is continuous at c and f is continuous at $g(x=c)$, then the composite function given by $f \circ g(x) = f(g(x))$ is continuous at c
 - $\lim_{x \rightarrow c} f(g(x))$
 - $f(x)$ is continuous at $g(c)$
 - $g(x)$ is continuous at $x = c$
- **Intermediate Value Theorem:** If f is continuous on the closed interval $[a,b]$ and k is any value between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c) = k$
 - $f(a) < k < f(b)$
 - $f(a) > k > f(b)$



Infinite Limits

- **Infinite Limits:** Let f be a function that is defined at every real number in some open interval containing c (except possibly at c). An “infinite limit” means that either
 - $\lim_{x \rightarrow c} f(x) = \infty$ of $\lim_{x \rightarrow c} f(x) = -\infty$
- **Vertical Asymptote:** The line $x = c$ is a vertical asymptote of the graph of f if $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left

- $\lim_{x \rightarrow c^-} f(x) = \pm\infty$

- $\lim_{x \rightarrow c^+} f(x) = \pm\infty$

● Properties of Infinite Limits

- Let b and L be real numbers and let f and g be function with the following limits

■ $\lim_{x \rightarrow c} f(x) = \infty$ of $\lim_{x \rightarrow c} g(x) = L$

- **Sum of difference:** $\lim_{x \rightarrow c} [f(x) + g(x)] = \infty$

- **Product:** $\lim_{x \rightarrow c} [f(x)g(x)] = \infty$ $L > 0$

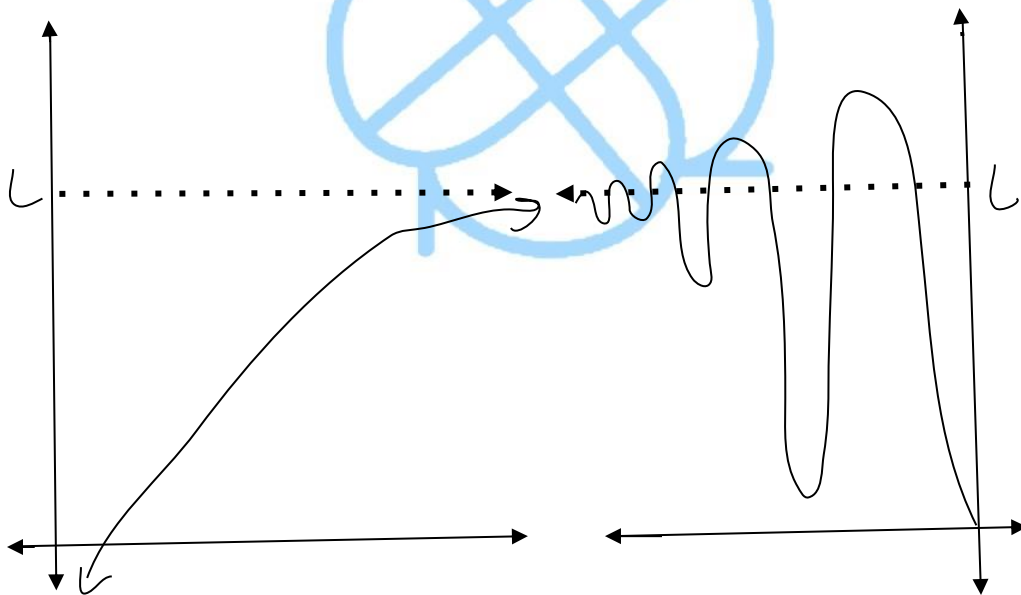
- **Product:** $\lim_{x \rightarrow c} [f(x)g(x)] = -\infty$ $L < 0$

- **Quotient:** $\lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0$ $\frac{L}{\infty}$

Limits at Infinity

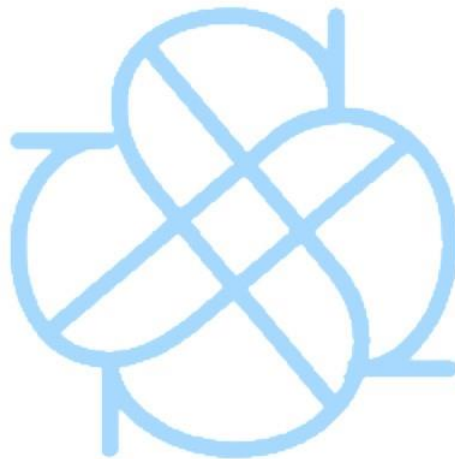
- $\lim_{x \rightarrow \infty} f(x) = L$

$\lim_{x \rightarrow -\infty} f(x) = L$



- **Definition:** The line $y = L$ is a horizontal asymptote of the graph f if

- $\lim_{x \rightarrow \infty} f(x) = L$ of $\lim_{x \rightarrow -\infty} f(x) = L$



Unit 10

Differentiation

- **Definition of the Derivative of a Function**

- $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

- **Slope** = $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{x+\Delta x - x}$
- **Derivative Notations:**
 - $f'(x)$ = first derivative of f
 - $\frac{dy}{dx} = \frac{\text{dependent variable}}{\text{independent variable}}$ = The derivative of y with respect to x
 - y' = "y" prime
 - $\frac{d}{dx}[f(x)]$ = take the derivative of $f(x)$ with respect to x
- **Example:** Find the derivative of $f(x) = x^2$

Handwritten derivation of the derivative of $f(x) = x^2$ using the limit definition:

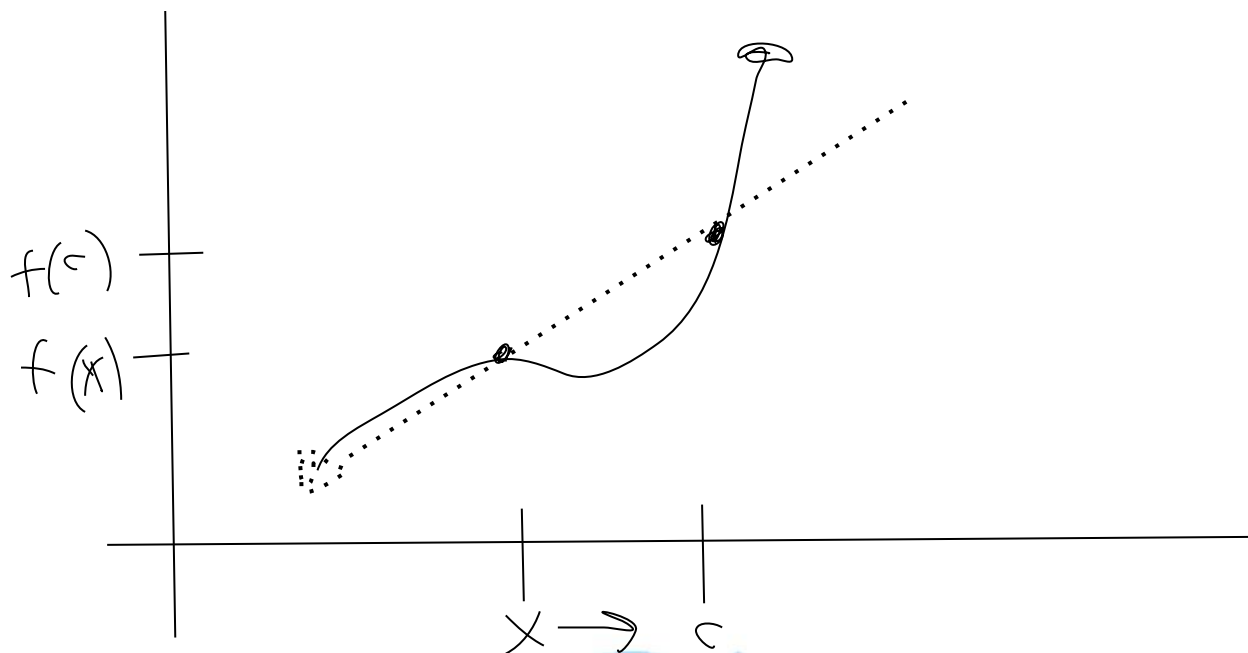
$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} \quad \text{D.S. } f'(x) = 2x + 0$$

$$\boxed{f'(x) = 2x}$$

- **Alternative form of the derivative (at a given point) $(c, f(c))$**
 - $f'(x) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$



- Example: Find the derivative with respect to t for the function $y = \frac{2}{t}$, when $t = 2$

$$y' \Big|_{t=2} = \lim_{t \rightarrow 2} \frac{\frac{2}{t} - \frac{2}{2}}{t - 2} = \lim_{t \rightarrow 2} \frac{\frac{2-t}{t}}{t-2} = \lim_{t \rightarrow 2} \frac{2-t}{t(t-2)} = \lim_{t \rightarrow 2} \frac{-1(\cancel{t-2})}{t(\cancel{t-2})} = \lim_{t \rightarrow 2} -\frac{1}{t}$$

Ans $\boxed{-\frac{1}{2}}$

- **Differentiability and Continuity**
 - Theorem: If f is differentiable at $x = c$, then f is continuous at $x = c$
- **4 Equations for differentiable equations**
 1. *Not continuous*
 2. *Sharp Turns/Cusps (absolute value)*
 3. *Vertical tangent line*
 4. *End Points*

Basic Differentiation Rules

- The Constant Rule

- $f'(x) = 0, c = \#$

- Power Rule

- $f'(x^n) = nx^{n-1}$

- Constant Multiple Rule

- $f'(cf(x)) = cf'(x)$

- Sum and Difference Rules

- $f'[f(x) \pm g(x)] = f'(x) \pm g'(x)$

- Product Rule

- $f'[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$

- Quotient Rule

- $f'(x)\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

- Chain Rule

- $f'[f(g(x))] = f'(g(x)) \cdot g'(x)$

- Derivatives of Trigonometric Functions

- $\frac{d}{dx} \sin x = \cos x$

- $\frac{d}{dx} \cos x = -\sin x$

- $\frac{d}{dx} \tan x = \sec^2 x$

- $\frac{d}{dx} \cot x = -\csc^2 x$

- $\frac{d}{dx} \sec x = \sec x \tan x$

- $\frac{d}{dx} \csc x = -\csc x \cot x$

Rate of Change

- $rate = \frac{distance}{time}$

- $Average\ velocity = \frac{change\ in\ distance}{change\ in\ time}$

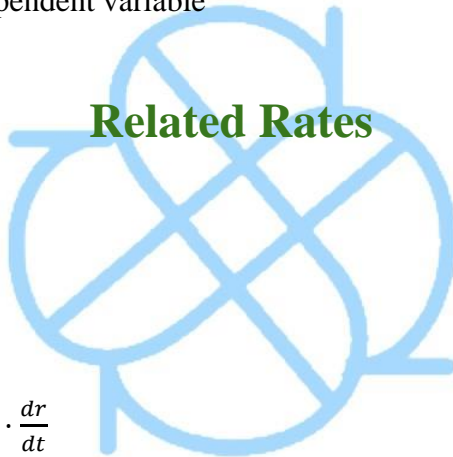
- **Position Function:** $s(t) = -\frac{1}{2}at^2 + v_0t + s_0$
 - a = acceleration (gravity)
 - v_0 = initial velocity
 - s_0 = Initial position
- The **sign** on velocities indicates the **direction**.
- **Instantaneous Velocity** is the velocity at a single point.
 - $s(t) = -\frac{1}{2}at^2 + v_0t + s_0$

Implicit Differentiation

- **Implicit:** Not solved for dependent variable
- **Explicit:** Solved for dependent variable

Related Rates

- Find $\frac{d}{dt}[x^2 + 3]$
 - $2x \cdot \frac{dx}{dt} + 0$
- Find $\frac{d}{dt}[\frac{\pi}{3}r^2h]$
 - $\frac{\pi}{3}r^2 \cdot \frac{dh}{dt} + \frac{2\pi}{3}rh \cdot \frac{dr}{dt}$



Works Cited

Desmos. "Desmos | Beautiful, Free Math." *Desmos.Com*, 2019, www.desmos.com/.

"Graphing Parabolas." *Saylordotorg.Github.Io*, saylordotorg.github.io/text_elementary-algebra/s12-05-graphing-parabolas.html#:~:text=Key%20Takeaways-. Accessed 23 Oct. 2020.

"Graphing Square Root Functions." *Www.Varsitytutors.Com*, www.varsitytutors.com/hotmath/hotmath_help/topics/graphing-square-root-functions. Accessed 23 Oct. 2020.

"How to Graph a Circle - Dummies." *Dummies*, 2016, www.dummies.com/education/math/calculus/how-to-graph-a-circle/.

McCalla, Jeff, and C.C. Edwards. "Convert between Degrees and DMS with the TI-84 Plus Calculator." *Dummies*, www.dummies.com/education/graphing-calculators/convert-between-degrees-and-dms-with-the-ti-84-plus-calculator/. Accessed 23 Oct. 2020.

Tiger. "Mr. Tiger Calculus." *Mr. Tiger Calculus*, mrtigercalculus.weebly.com/. Accessed 23 Oct. 2020.

VLB10525. "Parent Functions." *Slideshare*, 28 Sept. 2008, www.slideshare.net/VLB10525/parent-functions-presentation. Accessed 23 Oct. 2020.

Zegarelli, Mark. "How to Graph an Ellipse." *Dummies*,
www.dummies.com/education/math/calculus/how-to-graph-an-ellipse/. Accessed 23 Oct.
2020.

**** Some theorems and definitions are present in this study guide, that are not my own words**

