AP CALCULUS BC SIMPLE STUDIES REVIEW

Unit 1: Limits and Continuity

- **Limit**: the value that a function (typically denoted as 'f(x)') approaches as it gets closer and closer to a certain 'x' value
  - Ask yourself: "As 'x' approaches a value, what is f(x) getting closer to?
  - Typically, to find the limit of a function, you can just plug in 'x' to f(x)
    - Ex. What is the $\lim_{x \to 2}$ of x^2?
      - All you do is plug x (which = 2) into f(x) (which is x^2)
        - $2^2 = 4 = \lim_{x \to 2} x^2$
- **Left hand limit**: this is the value of f(x) as it approaches the value of x from the LEFT side
  - Denoted as: $\lim_{x \to 2^-}$
    - The '-' superscript indicates a LEFT HAND LIMIT
    - To find the answer, determine what the value that f(x) approaches as x = 1.9, 1.99, 1.999, etc.
      - Notice how the x values are less than 2, but are getting closer and closer to the actual value of 2.
- **Right hand limit**: this is the value of f(x) as it approaches a value of x from the RIGHT side.
  - Denoted as $\lim_{x \to 2^+}$
    - The '+' superscript indicates a RIGHT HAND LIMIT.
    - To find the answer, determine what the value that f(x) approaches as x = 2.1, 2.01, 2.001, etc.
      - notice how the x values are greater than two, but are getting closer and closer to the actual value of 2
- In order for a limit to exist, the LH and RH limits have to be equal to each other. If not, the limit does not exist (indicated as DNE).

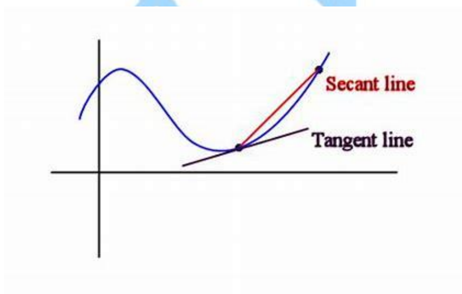
- Ex. $\lim_{x \to 0} 1/x$
    - The LH limit = '-∞' but the RH limit = '+∞'. Thus lim x→ 0 of 1/x = DNE
  - *REMEMBER: If the limit of a function equals '+∞', that doesn't mean that it doesn't exist. As long as both the RH and LH limits equal '+∞', then the limit can and does exist.
  - If given a graph, you can determine if the limit exists if there aren't any breaks in the graphs.
- If a question asks for the limit of f(x) as x→ '∞', this is just asking "what value does f(x) approach as x gets bigger and bigger.
  - Ex. lim x→ ∞ of 1/x
    - As you plug in greater values of x (x value gets larger and larger), the function gets closer and closer to 0, because 1/(a bigger number) gets smaller (closer to 0). Thus, the answer would be 0.
- When a function is a polynomial divided by another polynomial and x→ ∞ :
  - If the highest power of x for the numerator is greater than the denominator, then the limit of that function as x→ ∞ is ∞
  - If the highest power of x for both the numerator and denominator are equal, then the limit of that function as x→ ∞ is the coefficient of the highest power of x in the numerator divided by the coefficient of the highest power of x in the denominator.
  - If the highest power of x for the numerator is less than the denominator, then the limit of that function as x→ ∞ is 0.
- **Continuity**: A function is continuous ONLY if the LH limit = RH limit = f(x)
  - The limit of a function has to exist, and it has to equal the value of the function at that 'x' value.

        - Ex. lim x→ *2*of x^2= 4 (the limit exists because the RH limit and the LH limit are equal) and f(2) = 4. The limit x→ *2* and f(2) are both equal to 4, so the function is CONTINUOUS.
    - If given a graph, you can determine if it's continuous if there are no breaks in the graph, or open circles on that function on the graph.
        - Basically, the function on the graph looks continuous, with no holes in it.

Unit 2: Differentiation - Definition and Fundamental Properties

- **Differentiation**: Finding the derivative, or rate of change, of a function

- **Average rate of change** : The average rate of change of one quantity changing in respect to another
- **Instantaneous rate of change** : The rate of change at a specific moment/point
- **Derivative**: In simple terms, the derivative of a function is the slope of the secant line of that function, at a certain 'x' value.
    - Notation: f'(x), dy/dx, d/dx
    - Secant line: a line between two points
        - Equation of a secant line (AKA difference quotient) : $\frac{f(x+h)-f(x)}{h}$
            - Notice how this is basically the slope formula, except that x_1 and x_2 are replaced with x and h. 'H' just refers to the difference in the two 'x' values
    - Tangent line: a line that touches ONE 'x' point on the curve



        - To find the derivative of the tangent line, find the equation of the line that's tangent to the curve at the specified point

- Definition of Derivative:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

    - Ex. Find the derivative of f(x)=3x at x=2
        - Lim h→ 0 [3(x+h)-3(x)]/h = [3x+3h-3x]/h = 3h/h = 3
            - Lim h→ 0 of 3 = 3
            - Thus, f'(2) = 3
- Rules for a function to be differentiable:

        - A function HAS to be continuous
        - It CANNOT have a sharp corner (absolute value graphs are an example)
        - If the tangent line at a point in the function is VERTICAL, then it's NOT differentiable at that point

- **Power Rule**:

$$\frac{d}{dx}x^n = n \cdot x^{n-1}$$

    - Much faster way to determine the derivative of a function
    - Ex. Find the derivative of f(x) = $3x^2$, at x=2
        - according to the power rule, n=2, so f'(x)= 3(2)x^{2-1}which equals 6x.
            - Then, plug x=2 into 6x, which = 12. Thus, f'(2)=12
    - *the derivative of a constant is ALWAYS 0, because n=0 for all constants (it's implied)
- **Addition/Subtraction Rule for Derivatives:**
    - To find the derivative of a function that consists of adding/subtracting two functions, find the derivatives of each function separately, and then combine them.
        - Ex. Find the derivative of $3x^2$+ $2x^8$.
            - f'(x) of $3x^2$= 6x ; g'(x) of $2x^8$= 16x^7
            - Thus, the derivative of $3x^2$+ $2x^8$ is 6x + 16x^7
- You can keep on finding the second, third, fourth, etc. derivatives of a function too.
    - All you do is keep finding the derivative of that function
        - Ex. The first derivative of $2x^3$= $6x^2$
            - To find the second derivative, denoted as f''(x), just take the derivative of the first derivative, $6x^2$, which equals 12x
            - Continue to do this for the third derivative, etc.
- **Product Rule**:

$$\frac{d}{dx}(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

First f is the derivative; and then g is the derivative.

- Ex. Find the derivative of h(x) = ($2x^3$)($4x^2$-3x)
  - Let's set f(x)= $2x^3$ and g(x)= $4x^2$-3x
    - h'(x) = ($2x^3$)(8x-3) + ($4x^2$-3x)($6x^2$)
- **Quotient Rule:**
  - Remember: "[(low)(derivative of high) - (high)(derivative of low)]/low^2"
    - *High refers to the numerator, and low refers to denominator
    - Ex. Find derivative of f(x)=$5x^2$/($4x^2$-3x)
      - f'(x)= [($4x^2$-3x)(10x)-($5x^2$)(8x-3)] / $(4x^2 - 3x)^2$
- Derivative of Trig and Inverse Trig functions (it's best to just memorize all of these):

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}},\ x\neq\pm1$$

$$\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}},\ x\neq\pm1$$

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}},\ x\neq\pm1,0$$

$$\frac{d}{dx}\left(\csc^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}},\ x\neq\pm1,0$$

- **Derivative of ln(x) and** e^x(again, i'd just memorize them for convenience) :
  - f(x) = lnx → f'(x) = 1/x
  - f(x) = e^x→ f'(x) = e^x

Unit 3: Differentiation - Composite, Implicit, and Inverse Functions

- **Higher Order Derivatives:**
    - If you get a fraction for a function, or a square root:
        - Ex. Find the derivative of 3/x
            - Rewrite the function so that you can apply the power rule to it
                - 3/x = $3x^{-1}$, and now it's easier to use the power rule
        - Ex. Find the derivative of √x (square root of x)
            - Rewrite → √x = $x^{1/2}$, and then use the power rule to find the derivative
- **Chain Rule:**
    - Used to find the derivative of composite functions
        - Composite functions : f(g(x))
    - Chain rule formula:

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

        - Ex. y= $4(x^2 - 3x)^2$
            - f(x) = $4x^2$; g(x) = x^2-3x
            - Using the formula, you'd get 8 (x^2- 3x)(2x-3), which is the answer
- **Graphing derivatives:**
    - When given a graph of f(x), any point where the tangent line at that point is horizontal, means that the derivative at that point is 0 (because slope = 0)
        - These typically occur at local maximums or minimums of the graph, and they are the zeroes of a f'(x) graph
    - To determine if the derivative of a function is negative or positive at a certain point or interval, just look at the slope of the graph. Whenever the slope is negative, that means the derivative is negative, and if it's positive, then the derivative is positive.
    - The second derivative measures concavity of the function. If you're given the graph of f(x), and the graph is concave up, meaning it has a 'u' shape to it, then the SECOND derivative is positive. If the graph is concave down, with an upside down 'u' shape, then the SECOND derivative is negative.

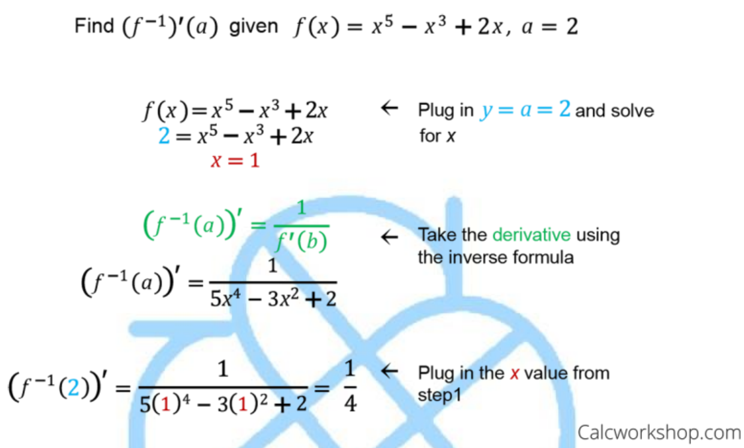
- To find the equation of a tangent line when given an equation, just take the derivative of the original function and plug in the given x value to determine the slope of the tangent line equation. If the y value wasn't already given for that specific x value, determine the y value by plugging the x value into the original equation. Then using the slope formula y = mx + b, you can figure out the equation of the tangent line.

- **Implicit Differentiation:**
  - When you can't isolate the function 'y' in terms of x, you need to use implicit differentiation.
    - Ex. Find dy/dx for the equation y^2-3y+2x = 3
      - To solve this problem, you tackle it the way you'd normally do, except whenever you take the derivative of any 'y' functions, you have to denote dy/dx too (look at next bullet point)
      - 2y(dy/dx) - 3(dy/dx) + 2 (dx/dx) = 0
        - *Remember, when taking the derivative of y in respect to x, you get dy/dx. Likewise, taking the derivative of x in respect to x will get you dx/dx, which is 1, and thus, it's usually implied and not written out.
        - Once you've taken the derivative of the whole equation, you just need to solve for dy/dx by isolating dy/dx
          - (dy/dx)(2y - 3) = -2 → dy/dx = -2/(2y - 3)
    - If the problem asks you to find the derivative of y in respect to t, you do the same thing, except for now it's dy/dt and dx/dt, and you now have to write dx/dt after you take the derivative of the x function too
      - Ex. 2y - 3x → derivative would be 2(dy/dt) - 3 (dx/dt)
  - To find the second derivative, take the derivative of the first derivative (after you isolated (dy/dx). Make sure you also apply the rules of implicit differentiation when you take the second derivative. Then, whenever plug in the original equation of dy/dx into your new second derivative to get your answer.
    - Using the example from above, dy/dx = -2/(2y - 3)

■ You need to use the quotient rule to find the second derivative of this equation → f''(x) = [(2y-3)(0)-(-2)(2(dy/dx))] / $(2y-3)^2$ → 4(dy/dx) / $(2y-3)^2$

- Next, plug in the original equation of dy/dx and simplify
    - f''(x) = 4(-2/(2y-3)) / $(2y-3)^2$→ f''(x) = (-8/(2y-3))/ $(2y-3)^2$→ f''(x) = -8/ $(2y-3)^3$

- **Derivative of an inverse function:**



- *Remember: to find the equation of the inverse function, you can plug in the given 'x' value into 'y' instead of 'x' as you'd normally do

Unit 4: Contextual Applications of Differentiation

- The derivative of a function determines the instantaneous rate of change with respect to its variable
    - Since the derivative of a function is basically the slope of that function at a certain point, the unit for the derivative is the unit of f(x) / unit of x
- **Straight line motion:**

    - The derivative of a position-time graph is its velocity, and the second derivative of a position-time graph is its acceleration. *note that speed is the absolute value of velocity
        - How did we derive this?
            - Well, the unit of the derivative is unit of f(x) / unit of x, and thus, when taking the derivative of position, the unit of the derivative would be m/s (or any unit of distance/unit of time), which gives you velocity. The second derivative of position is acceleration, which is the same thing as the first derivative of velocity, and by dividing the unit of fx) by the unit of x, the unit of the second derivative of position would be m/ s^2, which is the unit for acceleration.
- **Rates of change in other contexts:**
    - Similarly to motion, the rate of change is measured simply by taking the derivative of a function. Units are derived the same way.
- **Related Rates**: a problem that asks you to find the rate of change of a variable in relation to the rates of change of other variables
    - Ex. (taken from Princeton AP Calc BC Review book) : a circular pool of water is expanding at the rate of 16π in^2/ sec. At what rate is the radius expanding when the radius is 4 inches?
        - Steps to approach a related rates problem:
            - Set up an equation that connects the variables together.
                a. In this example, a circle's radius and the rate of change of area are given, so you can use the equation of the area of a circle, A = π r^2
            - Differentiate both sides of the chosen equation, with respect to time.
                a. dA/dt = 2πr(dr/dt)
            - Plug in the given values into the differentiated equation
                a. 16π in^2/ sec = 2π(4)(dr/dt)
            - Isolate variables to find the variable that you're looking for

a. dr/dt = $(16\pi \, in^2/\text{ sec}) / 8\pi = 2$ in/ sec

- **Linear approximation:**
    - You can use the equation of the tangent line of a graph to approximate the value of the function at a certain point.
    - Ex. Approximate f(2.1), when f(x) = x^2
        - First, find the equation of the tangent line, which is done by taking the derivative of the function at x =2 → f'(x) = 2x ; f'(2) = 4, and then using (x,y) to determine the equation. In this case (2,4) is a point on the graph.
            - Using y = mx + b, solve for b using (2,4), which gives you the final tangent line equation of y (or f'(x)) = 4x - 4
        - Once you have solved for the equation of the tangent line, plug in the value that the equation wants you to approximate into the equation
            - f'(2.1) = 4(2.1) - 4 = 4.4
    - To determine if the approximation is an overapproximation or an underapproximation, you need to know whether the original function is concave up or down. If the graph is concave up, then the linear approximation is an underapproximation, and if the graph is concave down, then the linear approximation is an overapproximation

- **L'Hospital's Rule:**
    - If the lim x → c gives you either 0/0 or ∞/∞, then you can use L'Hospital's rule to determine the limit

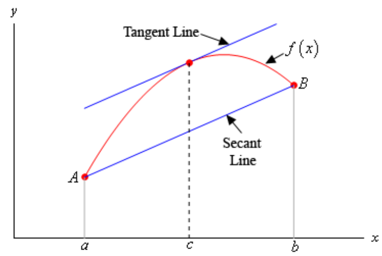
$$\text{If } \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\infty}{\infty}, \text{ then } \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

- Ex. $\lim_{x \to 2} (x^2 - 4) / (4x - 8)$ gives you 0/0, so you have to use L'Hospital's Rule, and basically just find the limit as x → 2 of the derivative of the numerator and the derivative of the denominator
  - $\lim_{x \to 2} (2x) / 4 = 4/4 = 1$

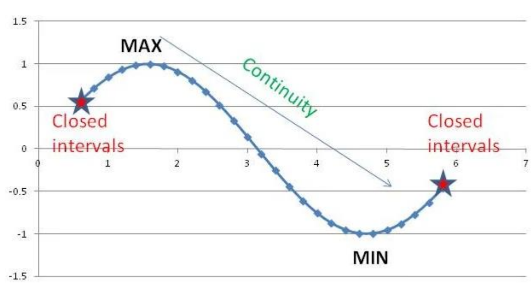
Unit 5: Analytical Applications to Differentiation

- **Mean Value Theorem (MVT)**: If a function is continuous over the closed interval [a, b] and differentiable over the open interval (a, b), then there's a point between 'a' and 'b' where the instantaneous rate of change equals the average rate of change over the interval.
  - $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- In terms of a graph, the MVT guarantees a point in which the tangent line of a graph at that specific point is equal to the slope between the endpoints of the graph.



- **Extreme Value Theorem**: If a function is continuous over a closed interval, then it must have a minimum and maximum value over that interval.
  - Minimum value: The smallest value of a function
  - Maximum value: the highest value of a function



- **Critical Point**: A point in a function where the first derivative at that 'x' value is equal to zero, or does not exist
  - To find the critical points of a function *when given an equation of the function*, find the first derivative of the function, and then determine the 'x' values where f'(x) = 0 or doesn't exist
  - To find the critical points of a function *when given a graph of the function*, determine the 'x' values where the tangent line at that point would be horizontal, indicating a first derivative of 0, or where the first derivative at that point is nonexistent, either because the tangent line is vertical, or because the graph at that point has a cusp, or sharp edge.
- **Local/Relative Extrema**: You say that a function has a local/relative maximum or a local/relative minimum at a certain 'x' value, which means that the graph at that point has a horizontal tangent line.

- *This doesn’t mean that the function has its highest/lowest value at that point (that’s the *absolute* maximum/minimum, which will be described later on. A local extremum can be an absolute max/min, but doesn’t have to)
- To find the local maximums and minimums of a graph:
  - 1) find the critical points of that function
  - 2) determine the concavity at each critical point of that function by taking the second derivative of the function.
    - If $f''(x) < 0$, then the function has a local maximum at that point
    - If $f''(x) > 0$, then the function has a local minimum at that point
- **note that all local extrema are critical points, but not all critical points are local extrema.
  - It’s possible for a critical point to have neither a local minimum or maximum, if $f''(x)$ doesn’t change concavity at that point.

- How to know if a function is increasing or decreasing:
  - A function is increasing if $f'(x) > 0$ at a certain point.
  - A function is decreasing if $f'(x) < 0$ at a certain point.
  - To determine the intervals in which a function is increasing/decreasing, find the critical points of a function, and then choose an ‘x’ value within an interval between 2 critical points or between a critical point and an endpoint, and plug ‘x’ into $f'(x)$. Then determine if $f'(x)$ is negative or positive, and that'll tell you whether the function is increasing or decreasing within that interval.
- **Absolute extrema**: the highest/smallest value of the function
  - Can only occur at critical points or endpoints
  - To find the absolute maximum or absolute minimum of a function:
    - 1) find the critical points of a function
    - 2) create a chart with one column for your ‘x’ values and one column for the corresponding f(x) value
    - 3) Fill in the ‘x’ values column with the endpoints of the function and all critical points
    - 4) solve for f(x) for each point in the ‘x’ values column, and the absolute maximum is the value of f(x) that’s the greatest out of all the other values,

and the absolute minimum is the value of f(x) that’s the smallest out of all the other values.

            - Absolute extrema are the ‘x’ points at which the absolute values occur

- **Concavity**:
    - As mentioned earlier, concavity of a function is determined by the second derivative.
        - If f’’(x) < 0, then the graph is concave down
        - if f’’(x) > 0, then the graph is concave up.
    - Concavity can also be measured by determining if the derivative of a function is increasing/decreasing at a point.
        - If f’(x) is increasing, then it’s concave up
        - If f’(x) is decreasing, then it’s concave down
    - Points of Inflection: a point of inflection is a point in a function where concavity changes at that point.
        - f’’(x) changes signs (either from positive to negative or negative to positive)
        - To find the point of inflection of a function, find the points where the second derivative is equal to zero. Then, determine the concavity of the intervals between the zeroes and the endpoints. If the second derivative changes signs between intervals, then there’s a point of inflection there.
- Connecting the function, its derivative, and second derivative (graphs)
    - As mentioned before, the derivative of a function is basically the slope of that function. Thus, whenever the derivative is positive, that means the slope of the function is positive, and the function is increasing. Likewise, if the derivative is negative, that means the slope of the function is negative and the function is decreasing.
        - When given a graph of f’(x), you can relate this graph to f(x) and f’’(x). On whatever intervals f’(x) is ABOVE the x-axis, that means the f(x) has a POSITIVE slope and is increasing on those intervals. Similarly, if f’(x) is BELOW the x-axis, then you know that f(x) is decreasing and has a

NEGATIVE slope on those intervals. The graph of f'(x) can also tell you the key features of f''(x), in the same way that the graph of f(x) tells you key features of f'(x).

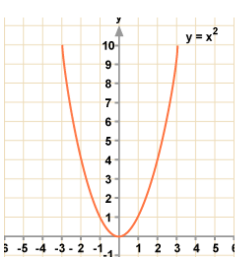
            - Basically, if f'(x) has a positive slope in an interval, that means that the graph of f''(x) should be above the x-axis in that interval.

        - The graphs of any derivative of a function can be related to each other.

- **Optimization Problems**: problems that ask you to find the "optimal", or best, value of a quality
    - The derivative of a function can be used to find the optimal value.
    - Steps to solving optimization problems:
        - 1) Determine the equation for the function that needs to be optimized.
        - 2) Express the function with one variable, by using substitution based off of the information given to you by the problem.
        - 3) Take the derivative of the function and set it equal to 0.
        - 4) Solve for the variable to determine the optimal value of the function.
    - Ex. Find the maximum area of a rectangle whose perimeter is 50 m^2 .
        - 1) Area = Length x Width → A = LW
        - 2) 2W = 50 – 2L → W = 25 – L → A = L(25–L) = 25L – L^2
        - 3) dA/dL = 25 – 2L → 0 = 25 – 2L
        - 4) L = 25/2
        - 5) W = 25 – L → W = 25 – 25/2 = 25/2 ; A = LW → A = (25/2)(25/2) = 625/4 m^2
            - *note how the problem asks you to find the maximum AREA. steps 1-4 only solved for the value of the LENGTH of the rectangle. The rest of the steps vary depending on what the problem's asking.

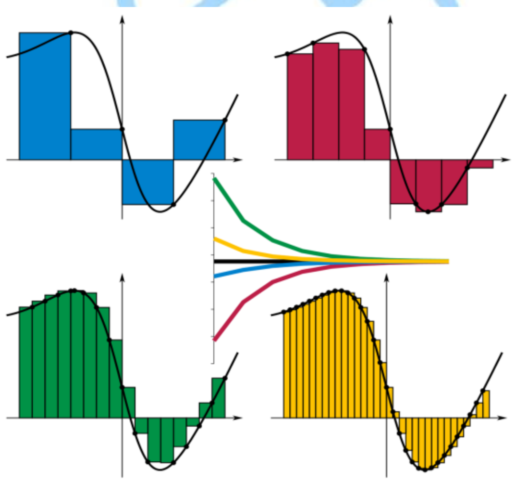
Unit 6: Integration and Accumulation of Change

- The area between a function on a graph and the x-axis can give us the accumulation of change.
  - For example, finding the area between a function that gives an object's velocity with respect to it's time and the x-axis would give us the total change in position.
  - Sometimes, you can just find the area by using basic geometry, if the area of the function is a square, a rectangle, etc.
    - Keep in mind that if the graph of the function is below the x-axis, then the area would be negative as well. If the graph is above the x-axis, then the area would be positive.
  - However, most of the time, the graph of the function is not an ideal shape, in which basic geometry cannot be used to find the area of the curve, such as the graph below:



  - The units of accumulation of change are found by multiplying the units of the independent variable to the units of the dependent variable.

- **Integrals**: The integral of a function is basically the ANTIDERIVATIVE of the given function.
    - The symbol to denote that you need to find the integral is: $\int$
    - Basically, if you are asked to find $\int 2x\, dx$, then you're trying to find a function that when you take the derivative of it, you get 2x.
        - "Dx" just represents the change in x, since you normally find the area by multiplying the width and the length (in simple terms). The width of the graph is the change in x, or "dx". The "length" of the graph is typically going to be the equation of the function, which in this case, is 2x (this will be explained more in depth later).
- **Riemann Sums (RS)**: an approximation of an integral that's used when the function isn't given
    - The approximation is made by adding the areas of a certain number of rectangles within the bounded graph (look at the picture below)



        - Look at the picture above. The areas of each graph can be approximated by adding up the areas of all the rectangles that are drawn into the graph. The height of the rectangles can vary depending on the type of Riemann Sum asked for.
            - In addition, notice how the greater the number of rectangles there are, the more accurate the approximation of the area of the function

bound between the function curve and the x-axis is (you know this because you see less white space showing in the graph with the yellow rectangles, than the one with blue rectangles.

- Riemann Sums are usually asked for when the problem doesn't give you the equation of the function, and instead just gives you values of the corresponding f(x) values to certain x values (typically give you a table)
  - Ex. Find $\int_2^8 f(x)dx$ using a [right/midpoint/left/trapezoidal] Riemann sum with 3 subintervals.

x	2	4	6	8
f(x)	3	16	21	12

    - The problem would specify the type of Riemann sum they're looking for and give you a table of values for you to use.
- There are 4 types: left, right, midpoint, and trapezoidal.
  - The **trapezoidal Riemann Sum** is the most accurate, then the midpoint RS, and then the left and right Riemann sums are the least accurate.
  - Right RS, Left RS, and Midpoint RS:

Right Riemann:

$$A = \frac{b-a}{n}[f(x_1) + f(x_2) + f(x_3) \ldots f(x_n)]$$

Left Riemann:

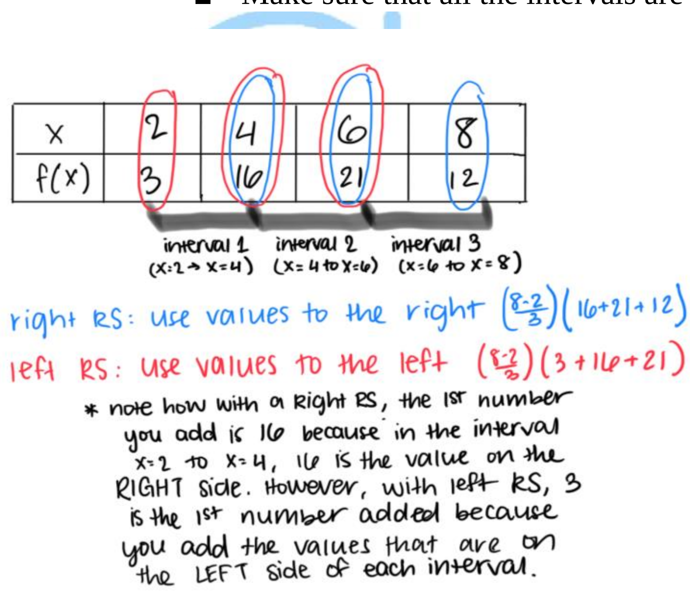
$$A = \frac{b-a}{n}[f(x_0) + f(x_1) + f(x_2) + f(x_3) + \ldots f(x_{n-1})]$$

Midpoint Riemann:

$$A = \frac{b-a}{n}[f(x_{1/2}) + f(x_{3/2}) + f(x_{5/2}) + f(x_{7/2}) + \ldots f(x_{n-1/2})]$$

    - These three RS are approached similarly; the only difference would be the values that are added (the ones in the brackets)
    - *remember that integrals are written in the form of $\int_{\square}^{\square} f(x)dx$. Thus, in the equation above, 'b' represents the upper limit of the integral, and 'a' represents the lower limit.
      - In the example above, b=8 and a=2.

- ‘N’ represents the number of subintervals that the questions asks you to divide the curve up into. Remember from the picture about Riemann sums that you can break up the curve into a lot of subintervals (so a lot of rectangular areas to add up), or even just a couple of subintervals only.
    - In the example above, n=3
- After recognizing the amount of subintervals wanted, break the table up into the correct amount of intervals.
    - In the example above, because the question asks for 3 subintervals, it’d only make sense to break the table up into intervals from x=2 to x=4, x=4 to x=6, and x=6 to x=8
        - Make sure that all the intervals are evenly spaced!



- The picture above explains which values to use for the right and left RS.
- For midpoint RS, the approach is, again, very similar to that of right and left RS, but the values used are different.
    - Midpoint means “middle”. Therefore, the values that are added up to find midpoint RS are the middle values in each interval.

      - In the example above, x=3 is the midpoint of the interval between x=2 and x=4. To find f(3), you just average the values of f(2) and f(4), so f(3) = 19/2. Then add f(3) to f(5) and f(7).
- Note that the formulas for RS above only work if the intervals are EVEN. In this case, each interval had a width of 2.

x	2	3	7	8
f(x)	3	19	25	12

- However, there are cases when they give you a table such as this one, where the intervals don't have the same width.
   - In this case, you'd still divide the table up into the 3 intervals (2,3), (3,7), (7,8), except the intervals aren't evenly spaced anymore.
   - Now, you'd have to subtract the endpoints of each interval to account for the different widths.
      - (1/3) [(3–2)(19) + (7–3)(25) + (8–7)(12) → this is for RIGHT RS
         - Notice how instead of multiplying all the added values to (b-a)/n, you have to multiply each individual value by the width of the interval.

- **Trapezoidal RS:**

Trapezoidal Method

- After creating a bunch of trapezoids, the area for the ith trapezoid is

$$\left[\frac{f(x_{i-1})+f(x_i)}{2}\right]\left(\frac{b-a}{n}\right)$$

- Which implies that

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}[f(x_0)+2f(x_1)+2f(x_2)+...+2f(x_{n-1})+f(x_n)]$$

Notice that the coefficients of successive terms follow the pattern of 1 2 2 2 2...2 2 1

	x_0	x_1	x_2	x_3
x	2	4	6	8
f(x)	3	16	21	12

Trapezoidal RS:

$$\int_2^8 f(x)dx = \frac{8-2}{2(3)}\left[3+2(16)+2(21)+12\right]$$

- *Take note that the formula above only works if the widths of the trapezoids are equal. However, if a question gives you a table and when you have divided up the values into the required number of "sections"

- **Approximation of Riemann Sums**
  - If a function is concave up, and a right RS is used, then the answer you get is an overapproximation of the area under the curve. If a left RS is used to find the approximation of the area under the curve of a concave up function, then it'd be an underapproximation.
  - If a function is concave down, then the right RS would be an underapproximation, and the left RS would be an overapproximation.

- The definite integral of a continuous function over a closed interval [a,b] is equal to the limit of Riemann sums as the width of the subintervals approaches 0 (indicated by the formula below)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\cdot\Delta x$$

$$\Downarrow$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + \left(\frac{b-a}{n}\right)k\right)\cdot\left(\frac{b-a}{n}\right)$$

- **Power rule for Antiderivatives**

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

  - EXCEPTION: when n= -1
    - *Remember that the derivative of f(x) = ln(x) is f'(x) = 1/x
      - Thus, ∫1/x dx = ln(x) + C
  - Ex. $\int x^4$dx = x^{4+1}/ (4+1) + C = $x^5/5$+ C
    - And if you take the derivative of $x^5/5$ + C, you'd get x^4
    - *'C' refers to any constant. Remember that the derivative of a constant is 0, so when taking the antiderivative of a function, you need to add 'C' to what you get.
- **The Fundamental Theorems of Calculus**

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$\text{where } F'(x) = f(x)$$

    - Ex. $\int_1^4$ ■ x^4dx = F(4) - F(1)
      - 1st, find the antiderivative of the function, which as we saw above, equals $x^5/5$ + C. Because this is a DEFINITE integral (bounded by x values), the "c" isn't necessary.

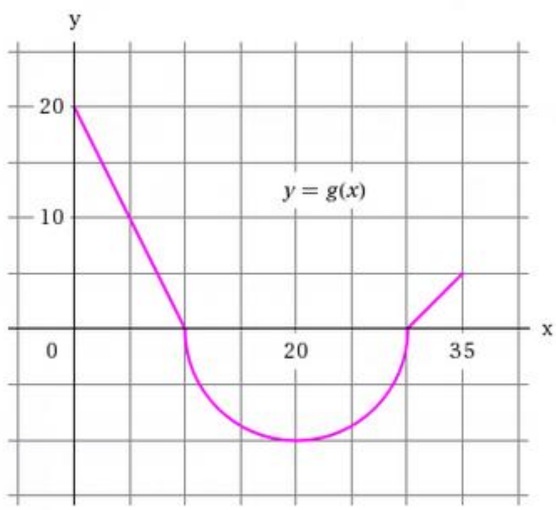
- Now that you know that F(x) = $x^5/5$, solve for F(4) - F(1), which equals $4^5/5$- ⅕ = 1023/5

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

- Ex. if g(x) = $\int_1^x t^4$dt, find g'(x) at x = 2.
  - According to the Fundamental Theorem of calculus, g'(x) = (d/dx) $\int_1^x t^4$dt = x^4. Then just plug x=2 into your equation, and you will get g'(2) = 16

- **Graphs and definite integrals:**

The graph of g consists of two straight lines and a semicircle. Use it to evaluate each integral.



- Let's say that g(x) = f'(x). Find f(35) if f(0) = 20.
  - In order to find f(35), you need to find the integral $\int_0^{35} f'(x)$dx, which according to the Fundamental Theorem of Calculus, should equal to f(35)-f(0) = $\int_0^{35} f'(x)$dx. Isolate f(35), and you'd get f(35) = f(0) + $\int_0^{35} f'(x)$dx = 20 + $\int_0^{35} f'(x)$dx.
    - Since we don't know what the equation of the function is, you can't use power rule to antidifferentiate and solve for $\int_0^{35} f'(x)$dx. However, we're given the graph of f'(x).

Remember that finding the integral of a function can be done by finding the area between the curve, and the x-axis.

- Thus, $\int_0^{35} f'(x)dx$ = the area of the triangle from (0,10), the semicircle from (10,30), and the mini triangle from (30,35).
  - $\int_0^{35} f'(x)dx = 100 - 50\pi + 25/2$
  - $f(35) = 20 + 100 - 50\pi + 25/2 = 245/2 - 50\pi$

- **Properties of Definite Integrals:**
  - Constants → ex. $\int_0^2 2x^2 dx$
    - When asked to find the integral of a constant times a function, the integral can be reformatted to $2\int_0^2 x^2 dx$, with the constant outside of the integral, and solve the integral separately, before multiplying 2 to it.
  - Addition and subtraction → ex. $\int_0^2 2x^2 - x dx$
    - $\int_0^2 2x^2 - x dx$ is equal to $\int_0^2 2x^2 dx - \int_0^2 x dx$
      - You can integrate each function independently, and then add/subtract as needed
  - Reversal of limits of integration
    - Typically when you see an integral, the lower limit is less than the higher limit. Ex. $\int_0^2 f(x)$→ 0 < 2, and so 0 is placed at the bottom.
      - However, sometimes, an integral can be written like this: $\int_2^0 f(x)dx$. In this case, you can reverse the limits, so that it's written with the lower number on the bottom, but when you reverse them, you need to negate the function.
        - Ex. if $\int_2^0 f(x)\,dx = 5$, then $\int_0^2 f(x)dx = -5$

- **Integrals of Trig Functions:**
  - Again, it's easiest to memorize the formulas for the integral of trig functions.

$\int \sin x \, dx$	$-\cos x + C$
$\int \cos x \, dx$	$\sin x + C$
$\int \tan x \, dx$	$-\ln(\cos x) + C$
$\int \cot x \, dx$	$\ln(\sin x) + C$
$\int \sec x \, dx$	$\ln(\sec x + \tan x) + C$
$\int \csc x \, dx$	$-\ln(\csc x + \cot x) + C$
$\int \sec^2 x \, dx$	$\tan x + C$
$\int \csc^2 x \, dx$	$-\cot x + C$

- **U-Substitution:**
    - Use u-substitution to reverse a function that was differentiated by chain rule.

$$\int f(g(x))g'(x)dx, \quad \text{if } u = g(x)$$

$$du = g'(x)dx$$

$$\text{then, } dx = \frac{du}{g'(x)}$$

$$\int f(u)g'(x)\frac{du}{g'(x)}$$

$$\int f(u)du$$

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- This is an overview of how u-substitution works. The example below will show you how to implement u-substitution

find $\int_3^5 2x\sqrt{x^2 - 9}\,dx$

- put $u = x^2 - 9$
- $du = 2xdx$
- when $x = 3$, $u = 0$ when $x = 5$, $u = 16$

$$I = \int_0^{16} u^{1/2} du$$

$$I = \left(\tfrac{2}{3} u^{3/2}\right)_0^{16} = \frac{128}{3}$$

- Another way to solve this is to plug 'x^2 - 9' back into 'u' after integrating '$u^{1/2}$du' and keep the interval from x=3 to x=5.
  - Basically, the last step can look like this:
    - I = $\left[(2/3)(x^2 - 9)^{3/2}\right]_3^5$

- Ex. 2: $\int_0^{\pi/2}$ cosx(sinx)^{2}dx
  - In this case, u=sinx (usually, the function that has the higher exponent is the function you use for 'u')
    - Du = cos(x) dx, and you can rearrange this equation to get dx = du /cosx
  - Then, plug in your newly derived equation in dx, so that you get $\int_0^{\pi/2}$ (cosx)u^2(du/cosx)
    - Now you can cancel out the 'cosx', and are left with $\int_0^{\pi/2}$ u^{2}du.
  - Integrate, and you'd get $[u^3/3]_0^{\pi/2}$
    - *note that 0 and π/2 are the limits of the integral when the function was in terms of 'x', not 'u'.
      - To account for this, you can change the limits by plugging x=0 and x=π/2 into u=sinx, to find the limits for 'u'
        - When x=0, u=0 and when x=π/2, u=1
        - Now, replace the limits, and solve for $[u^3/3]_0^1$
      - The second way to account for this is to substitute the original function back into 'u'
        - You'd get: $[(sin^3x)/3]_0^{\pi/2}$. Now you can solve for the answer.

- **Integrating Functions using long division:**
  - You can use long division to simplify the function before integrating to make it easier.
  - Ex. $\int \frac{x^2+x+1}{x^2+1}$dx.
    - Use long division to simplify: $\int \frac{x}{x^2+1} + 1$dx, and now it's easier to integrate the function. You can use the additive rule to break up the

integral into v$\int \frac{x}{x^2+1}dx + \int 1$dx, and integrate the first integral using u-substitution and the second integral using power rule (or remember that the integral of any constant is that constant times x, so in this case 1(x) = x)

- **Integration by Parts:**
  - This is used to find the antiderivative of a function that was derived using the product rule.
  - Formula: $$\int udv = uv - \int vdu$$
  - Ex. (from the Princeton 2020 AP Calculus BC book): find $\int$ xsinx dx
    - 1) choose what function you want to be 'u' and 'dv'
      - Technically, you can choose any function to be 'u' and 'dv', but there are cases where it'd be a lot easier to assign a certain function to 'u'
        - LIATE rule: use this to determine what function should be 'u' → the function that comes earlier in the acronym "LIATE" should be the one assigned as 'u'
          - L : logarithmic functions (ex. log(x), ln(x))
          - I : inverse trig functions (ex. $\sin^{-1}(x)$, $\cos^{-1}(x)$)
          - A : algebraic functions (ex. x, $x^2, x^2 + 1$, 2x + 3)
          - T : trig functions (ex. sinx, cosx, tanx)
          - E : exponential functions (ex. $2^x, 3^x$)
      - In this ex., breaking up the integral's functions will give you x (algebraic function) and sinx (trig function)
        - Because 'A' comes before 'T' in the acronym "LIATE", u = x, which means that dv = sinx.
    - 2) draw a table with the values of u,du,v,dv on it
      - *this is optional, but it tends to help a lot, especially if you're having a hard time understanding this

- In this ex., the table would look like this:

u = x	v = -cosx
du = 1 dx	dv = sinx dx

- *note that because sinx = dv, you need to integrate that function to find the function for ‘v’

▪ 3) plug the functions in the table into the formula

- $\int$ xsinx dx = x(-cosx) - $\int$ -cosx dx

▪ 4) integrate the last part of the equation, and solve for the integral. If it’s a definite integral, follow the steps of normal integration to get a numerical answer.

- $\int$ xsinx dx = -xcosx + sinx + C

- Integration using partial fractions 1:
  - Used to solve for integrals that contain rational expressions.
    - Ex. of rational expression: $\int \frac{3x + 5}{(x-2)(x-1)}dx$
  - It’s important to first note that rational expressions can be split up into separate fractions like $\frac{A}{x-a}+\frac{B}{x-b}$

Example 1: Write $\frac{4x+1}{x^2-x-2}$ using partial fractions.

$$\frac{4x+1}{x^2-x-2} = \frac{4x+1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

.

▪ The picture above walks through the steps on how to split up the rational expression into 2 fractions that are added to each other.

- 1) set the rational expression equal to $\frac{A}{x-a}+\frac{B}{x-b}$ as shown above
  - (x-a) and (x-b) correlates to the denominator of the rational expression in the integral.
    - (x+1) is placed before (x-2) in the integral, so ‘a’ in ‘x-a’ = 1, and ‘b’ in ‘x-b’ = 2

            - 2) multiply both the numerator and denominator by the denominator of the rational expression that's being asked to be integrated (in the ex. Above, it's (x+1) and (x-2))
                - This should leave you with 4x + 1 = A(x-2) + B(x+1)
            - 3) You can reorganize the equation by distributing → 4x + 1 = Ax - 2A + Bx + B → 4x + 1 = x(A+B) + (-2A + B)
                - This makes it easier to solve for A and B
            - 4) Solve for A and B using elimination
                - In this example, A + B = 4 and -2A + B = 1
                - Using elimination, you'd eventually get A = 1 and B = 3
            - 5) Plug the values of A and B into the partial fraction and replace the rational expression in the integral with this new partial fraction → $\int \frac{1}{(x+1)} + \frac{3}{(x-2)}dx$
            - 6) Now you can easily integrate the function using the additive rule → $\int \frac{1}{(x+1)}dx + \int \frac{3}{(x-2)}dx$ = ln(x+1) + 3ln(x-2)
- Integration using partial fractions 2:
    - Used if the denominator has a repeated function
        - Ex. from Princeton 2020 AP Calculus BC prep book: $\int \frac{2x+4}{(x-1)^2}dx$
            - 1) Rather than setting the rational expression equal to $\frac{A}{x-a} + \frac{B}{x-b}$, instead, we set it equal to $\frac{A}{x-a} + \frac{B}{(x-a)^2}$
                - In this ex. $\frac{2x+4}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$
            - 2) solve for A and B using the same steps as before.
                - Multiply the whole equation by $(x-1)^2$ → 2x+4 = Ax + (B - A)
                - A = 2 because that's the only value of A that will make the equation equal. Because (B-A) = 4, B = 6

- 3) Plug in the values of A and B into the partial fraction equation, and then replace the rational expression with the new partial fraction equation.
- 4) Integrate using the additive rule

- Integration using partial fractions 3:
  - Use if the rational expression is in the form : $\frac{px^2 + qx + r}{(x-a)(x^2+b)}$
    - **Ex.** $\int \frac{2x+1}{(x-2)(x^2+1)}$dx
      - Now, you'd set this rational expression equal to $\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
        - $\frac{2x+1}{(x-2)(x^2+1)} = \frac{A}{(x-2)} + \frac{Bx+C}{(x^2+1)}$
          - *note that 'Bx+C' is the numerator for the fraction with the denominator of a quadratic equation, which is $(x^2 + 1)$in this example.
      - Solve for A, B, and C using the same method as before.
      - At the end, when replacing A, B, C into the partial fraction equation, make sure you plug in C too!
      - Integrate using the additive rule. You may have to break up the fraction $\frac{Bx+C}{(x^2+1)}$even further to integrate more easily.

- **Improper integrals:**
  - An integral is improper if one/both of the limits of the integral is infinite, or if the integrand becomes infinite at 1+ points in the interval of integration
    - Ex. $\int_0^{\infty} \frac{dx}{(x^2+1)}$(one of the limits is infinite) or $\int_0^{\pi/2}$ tanx dx (tan($\frac{\pi}{2}$)= ∞)
      - Ex. 1 : $\int_0^{\infty} \frac{dx}{(x^2+1)}$(example taken from Princeton book)
        - replace the upper limit with a, and take the limit of the integral as it approaches infinity → $\lim_{a\to\infty} \int_0^a \frac{dx}{(x^2+1)}$
        - integrate the function → $tan^{-1}(a) - tan^{-1}(0)= tan^{-1}(a)$

- Take the limit of the integrated function →

  $\lim_{a\to\infty} tan^{-1}(a) = \frac{\pi}{2}$

- Ex. 2 : $\int_0^{\square/2}$ tanx dx
  - Similar to ex.1, replace the upper limit with a, and take the limit of the integral as it approaches infinity →

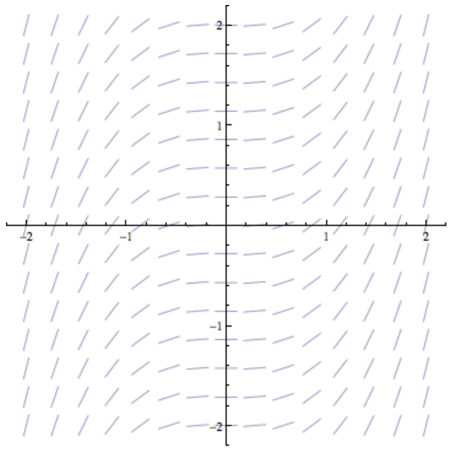
    $\lim_{a\to\frac{\pi}{2}} \int_0^{a}$ tanx dx

  - Integrate the function → $\ln|sec(a)| - \ln|sec(0)|$
  - Take the limit of the integrated function →

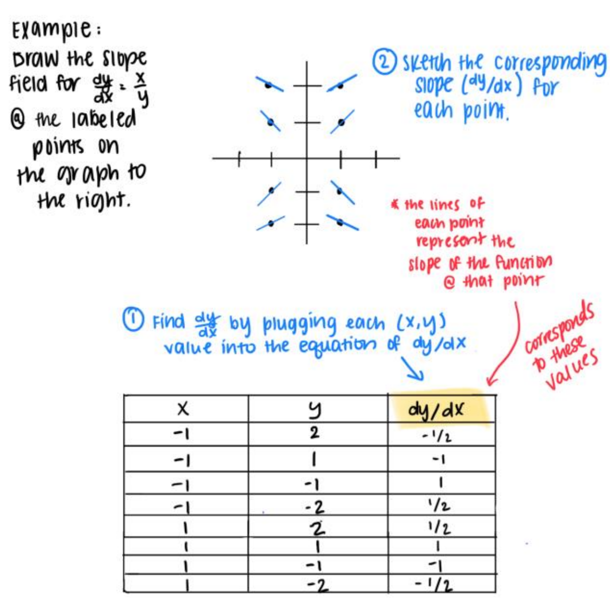
    $\lim_{a\to\frac{\pi}{2}} \ln|sec(a)| - \ln|sec(0)| = \infty - 0$ = undefined

Unit 7: Differential Equations

- **Separation of variables:**
  - When given the derivative equation and a function it's equal to, you can find the original function using separation of variables.
  - Ex. find the equation for y in terms of x if $\frac{dy}{dx} = 2xy$ and y(0) = 2
    - 1) rearrange the equation so that each variable is separated from each other, so in this case, have all 'y' variables on one side, and all 'x' variables on the other → $\frac{dy}{y}$= 2xdx
    - 2) integrate both sides of the integration → $\int \frac{dy}{y} = \int 2xdx = \ln|y| = x^2 + C$
    - 3) solve for C by plugging in the values the question provides (y(0) = 2) → $\ln|2| = C$
    - 4) isolate y to find the equation of y → y = $e^{x^2 + ln|2|}$
- **Slope fields**: graphical representation of a differential equation on a finite set of points in the plane
  - They basically tell you the derivative of a function at each point on a graph. The graph below is an example of a slope field.
    - Horizontal lines mean that the slope at that point is 0.
    - Vertical lines mean that the slope is undefined at the point.



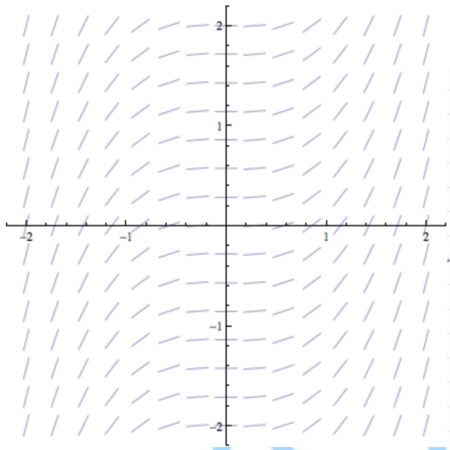
- How to sketch slope fields:
  - When asked to sketch the slope field of a function, you're given the differential equation for that function.
    - Ex. $\frac{dy}{dx} = \frac{x}{y}$
      - All you do is plug in points (x,y) into the given differential equation to find the slope, and then make a small hash mark (just like the ones in the graph above) at that point to demonstrate the slope.

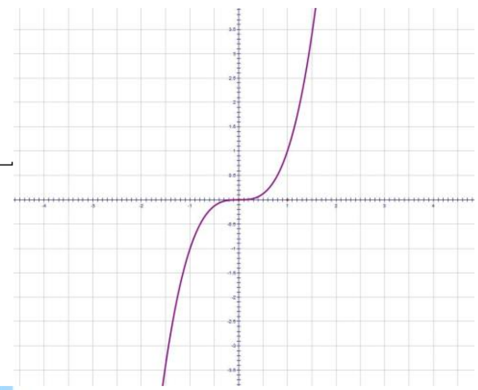


x	y	dy/dx
-1	2	-1/2
-1	1	-1
-1	-1	1
-1	-2	1/2
1	2	1/2
1	1	1
1	-1	-1
1	-2	-1/2

- A slope field can be used to guess the equation of the original function

- Many times, the shape of the slope field can kind of give an idea of the graph of the original function
  - For example, looking at the slope field below (pictured on the left), you can see that the shape of the original function is most likely a variation of y=x^3(pictured on the right)

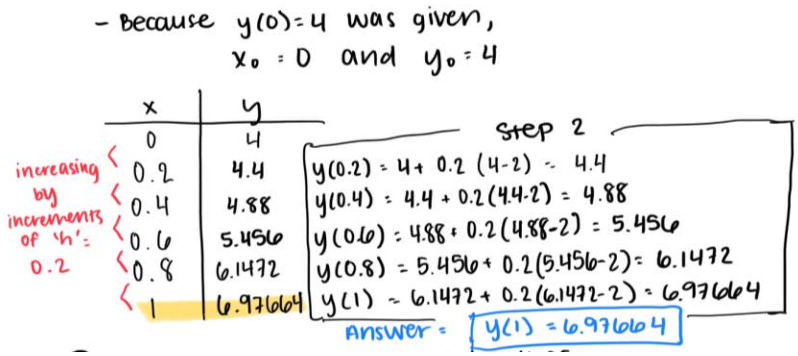




- **Euler's Method:**
  - Used to find an approximate solutions using differential equations
  - Formula: $Y_n = Y_{n-1} + hF(X_{n-1}, Y_{n-1})$, where 'F' = f'(x). 'H' = Δx
  - Example (from Princeton book): Use Euler's Method, with h = 0.2, to estimate y(1) if y' = y - 2 and y(0) = 4

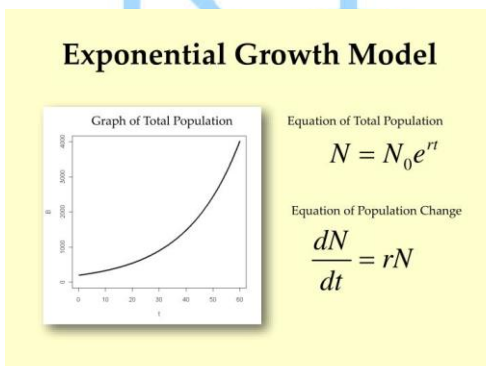
① Draw Table + fill in all the 'x' values
- you know what the 'x' values because the problem tells you that $x_0 = 0$, and that 'h', or $\Delta x = 0.2$. Keep on adding 0.2 to the 'x' values until reaching the value that the problem asks for.
  - ▸ In this case, it asks for y(1), meaning x=1

② Find out the corresponding 'y' values for each 'x' value using Euler's formula: $y_n = y_{n-1} + hf'(x_{n-1}, y_{n-1})$
- keep doing this until you reach the y(n) value that you're looking for

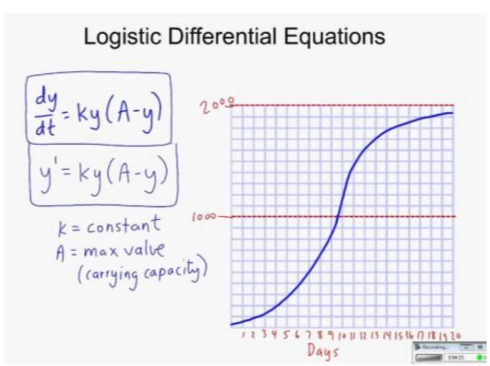


- *The important thing to remember with Euler's formula is that the equation asks for the PREVIOUS x and y values

- **Exponential growth:**
  - Exponential growth and decay model is $\frac{dy}{dt} = ky$
    - With the initial condition y = y_0 when t = 0, the exponential growth model has solutions of the form y = $y_0 e^{kt}$.



- **Logistic growth:**
  - This growth model is typically used with population



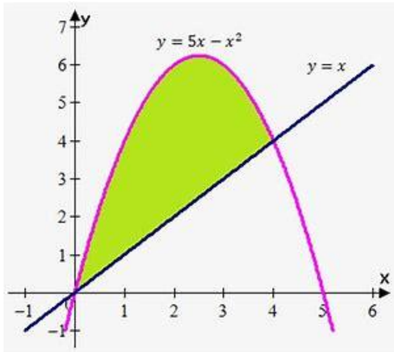
    - Carrying capacity refers to the maximum population that the environment can sustain; it's the limiting value

Unit 8: Applications of Integration

- **Average value:**
  - Used to find the average of all the instantaneous values of a function

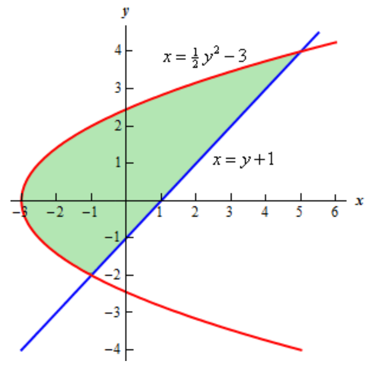
$$\frac{1}{b-a}\int_a^b f(x)dx$$

  -
    - Use the formula above to calculate average value.
  - Example: find the average value of f(x) = x^3 from x = 1 to x = 3
    - All you do is plug in the values into the formula → $\frac{1}{3-1}\int_1^3 x^3 dx$
    - Then solve for the integral → (½)[$(3)^4/4$ - $(1)^4/4$]= 10
- **Connecting position, velocity, and acceleration**
  - The integral of acceleration represents an object's velocity over an interval of time
  - The integral of velocity represents an object's displacement over an interval of time.
  - The integral of speed represents the object's total distance travelled in that time interval
- **Applying integrals into context**
  - Remember that the integral of a function represents the accumulation of the rate of change (AKA net change) of that function over the time interval
- **Area between two curves as functions of x**



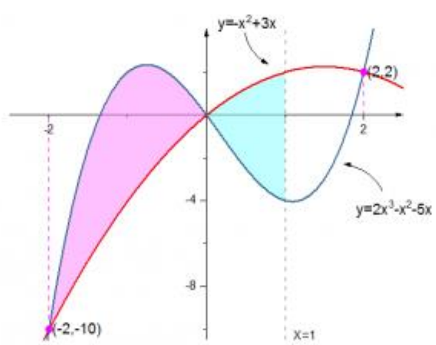
- 1) To find the area between the curves of two different functions in respect to x, it's easiest to sketch the two curves if possible. This just makes it easier to understand how to approach the problem.
- 2) Then, find the points of intersection between the curves, and this will determine the limits of your integral.
  - This can be done by setting the two equations equal to each other, since both equal 'y', and thus, they must be equal to each other → $5x - x^2 = x$, and then solve for x → $4x = x^2$ → x=0, x=4
    - Thus the limits of this integral are from x=0 to x=4
- 3) set up the integral → $\int_a^b \ [function\ that's\ "on\ top"] - [function\ that's\ "below"]dx$
  - In this example, $y = 5x - x^2$ is the function that's "on top", and y = x bounds the area at the bottom of the shaded area
    - $\int_0^4 \ 5x - x^2 - x\,dx = \int_0^4 \ 4x - x^2\,dx$
- 4) integrate, and solve for the area.

- **Area between curves as a function of y**
  - If the equation of the functions are given in terms of y, then the same steps are used to find the area between the curves, with some changes
    - The limits of the integral will change if the functions are in terms of y. Rather than setting the limits with x values, the limits will be set using y values.



- As you can see in the picture above, the functions are in terms of y. The limits of the integral would be from y = -2 to y = 4, because that's where the two curves intersect.
  - Also, when writing the integral, rather than being in terms of $\int_a^b f(x)\,dx$, it'll now be written as: $\int_a^b f(y)\,dy$, and to determine which function is written first in the formula, rather than the function "on top" going first, you now determine the function that's "to the right"
    - In this example, x= y + 1 is the function that's binding the right side of the area between the curves, so the integral would be set up like this: $\int_{-2}^{4} y + 1 - (\frac{1}{2}y^2 - 3)dy = \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\,dy$

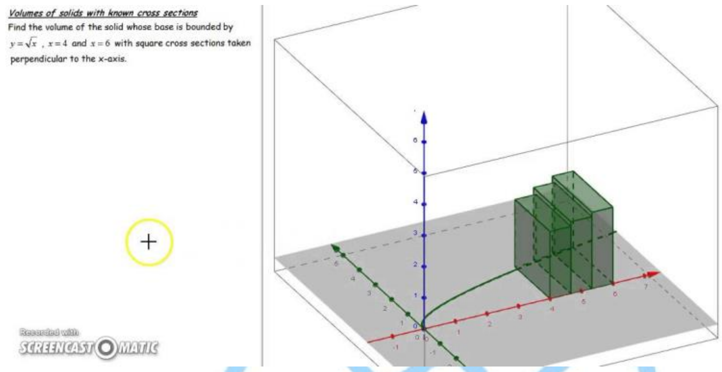
- **Area between curves with more than 2 intersection points.**
  - If there are more than 2 intersection points between two curves, then multiple integrals will need to be set up.



  - In the picture above, the curves have 3 intersection points. The blue curve $(2x^3 - x^2 - 5x)$is the curve on top from x = -2 to x = 0, but then it becomes the curve on the bottom from x = 0 to x = 2. Thus, 2 integrals need to be set up and added to each other to account for the change:

- $\int_{-2}^{0} \quad 2x^3 - x^2 - 5x - (x^2 + 3x)\, dx + \int_{0}^{2} \quad x^2 + 3x - (2x^3 - x^2 - 5x)\, dx$
    - There needs to be two integrals because the function that's "on the top" changes at x = 0, and thus, the integrals have to change, or else the wrong area would be found.

- **Volumes with cross sections:**

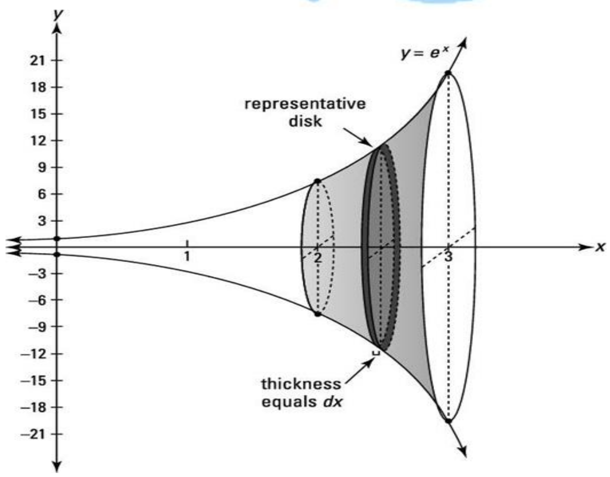


  - The picture above demonstrates what is meant by a "cross-section". Basically, finding the volumes of solids using cross sections is done by adding up numerous rectangles (or whatever shape the problem asks for) together.
  - Square cross sections:
    - $\int_{a}^{b} \quad (f(x))^2 dx$
      - The function is squared because the height of the volume is f(x), since squares have the same length for all 4 sides. Integrating this allows you to "add up" all of the square cross sections within the given interval.
    - For the problem above, the integral would be set up like this: $\int_{4}^{6} \quad (\sqrt{x})^2 dx = \int_{4}^{6} \quad x\, dx$. After setting up the integral, integrate to solve for the volume.
    - *note that in this example, it explicitly said that the cross sections were perpendicular to the x axis. Thus, the function was written in terms of x. However, there are problems that ask for the volume of a solid with cross

sections that are perpendicular to the y axis, in which case you'd need to write the function in terms of y.

- **Rectangular cross sections:**
  - $\int_a^b \quad f(x)h\,dx$
    - The only difference for a rectangular cross section is that because the height of the solid isn't the same as its base, you can't square the function. Instead, the height will be given, so you just multiply the height of the solid to the function.
  - For ex., if the problem above asked for rectangular cross sections, and said that the height of the solid was h=4, the integral would be set up like this: $\int_4^6 \quad 4\sqrt{x}\,dx$, and then you can integrate as you normally would.
- *If the problem gives you two curves, and asks you to find the volume of a solid who's base is the area between the two curves, the integral would be set up like: $\int_a^b \quad (f(x) - g(x))h\,dx$
  - Just like finding the area between two curves, f(x) is the curve that's "on top", or if it's in terms of y, "to the right"

- **Disk Method:**
  - This method is used to find the volume of solids whose cross sections are circular disks.

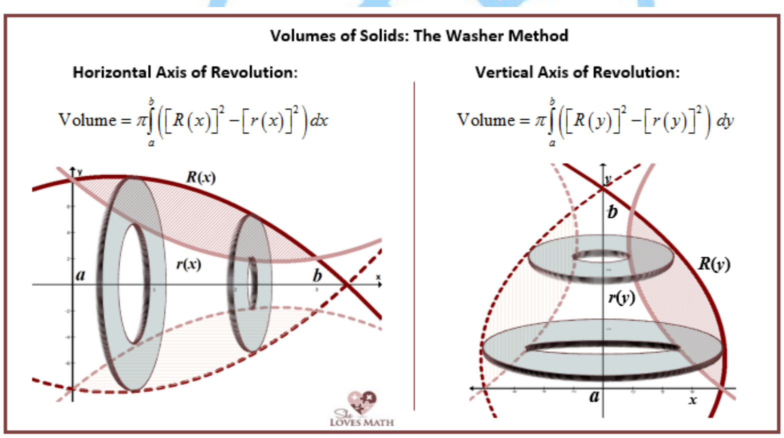


  - If you take the curve y=$e^x$and revolve it around the x-axis, then you can find the volume of the generated solid by adding up the volumes of all of the disks.

- 1) Each disk has an area of πr^2. To find the area of the disks, you can plug the equation of the function into 'r' → $\pi(e^x)^2$. The volume of the disk is equal to the area of the circle times the thickness of each disk, which is 'dx' → volume of each disk = $\pi(e^x)^2$dx
- 2) Then, to add up all of the volumes of each disk, integrate this equation using the given limits. In this case, the graph shows you that the limits of the solid are from x=0 to x=3 → $\int_0^3 \pi(e^x)^2 dx$.
- 3) Integrate the function to find the volume of the solid.

  - You can also use the Disk Method for a function that's being revolved around the y-axis. The only difference is that the function being integrated should be in terms of y.

- **Washer's Method:**
  - This is used to find the volume of a solid whose cross sections are ring-shaped (look at image below)

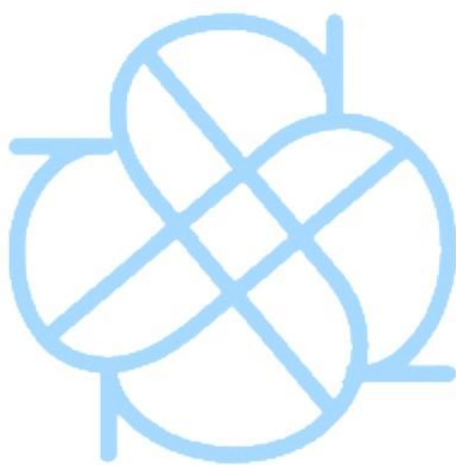


  - The formula for Washer's Method is in the picture above.
    - R(x) represents the equation of the curve that accounts for the outside of the ring, and r(x) represents the inner circle of the ring.

- **Arc Length / Length of a Curve:**

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

- If the function is continuous and differentiable on [a,b], then the formula above can be used to find arc length.
- Ex. (taken from Princeton book): 1: Find the length of the curve $y = x^{3/2}$ from x = 0 to x = 4.
  - 1) find f'(x) → f'(x) = $\frac{3}{2}x^{1/2}$
  - 2) plug in the values into the formula → $\int_0^4 \sqrt{1 + (\frac{3}{2}x^{1/2})^2}dx$
  - 3) integrate → $\int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$ (use u-substitution to finish integrating)
- Arc length of a planar curve can give you the distance travelled by an object.



Unit 9: Defining and Differentiating Parametric Equations

- **Parametric functions**: a pair of functions that allow you to find the coordinates of both x and y, usually written in terms of 't'
  - Parametric equations are usually written in terms of 't'.
- Calculating the derivative of parametric functions:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

- Ex. If $x = t^2$ and $y = t^3 + 1$, find $\frac{dy}{dx}$
  - 1) find dx/dt and dy/dt → dx/dt = 2t and dy/dt = $3t^2$
  - 2) plug the values into the formula → $\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3}{2}t$

- Calculating the second derivative of parametric equations:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

  - Using the example from above, to find the second derivative of that parametric function, take the derivative of $\frac{dy}{dx}$ and then divide that by $\frac{dx}{dt}$ → $\frac{d^2y}{dx^2} = \frac{(3/2)}{2t} = \frac{3}{4t}$

- **Finding the arc length of parametric curves:**
  - $$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
    - From t=α to t=β
  - Ex. (from Princeton book): Find the length of the curve defined by x=sin(t) and y=cos(t), from t=0 to t=π
    - 1) find (dx/dt) and (dy/dt) → $\frac{dx}{dt} = cos(t)$ and $\frac{dy}{dt} = -sin(t)$
    - 2) plug into the formula →

      L=$\int_0^{\pi} \sqrt{(cos^2t) + (sin^2t)}\, dt = \int_0^{\pi} \sqrt{1}\, dt = \int_0^{\pi} dt = \pi$

      - Remember that $(sin^2t) + (cos^2t) = 1$

- **Vectors:**
  - Vectors are quantities that have both a magnitude and direction
  - Vector functions are similar to parametric functions because they describe the motion of a particle on a plane

- Vector functions have similar relationships between position, velocity, and acceleration as we've seen earlier.
  - If a vector function P = position of a particle, then the derivative of P gives you velocity, and the derivative of velocity gives you the particle's acceleration
    - Both x(t) and y(t) have to be differentiable for the vector function to be differentiable as well
    - To find speed of a vector function, take the magnitude of the velocity vector function
      - The magnitude of a vector function = $\sqrt{x^2 + y^2}$
- Denotation of vectors:

This symbol indicates a vector function

position: $\vec{p}(t) = x(t)\hat{i} + y(t)\hat{j}$

velocity: $\vec{v}(t) = x'(t)\hat{i} + y'(t)\hat{j}$

acceleration: $\vec{a}(t) = x''(t)\hat{i} + y''(t)\hat{j}$

* $\hat{i}$ indicates left + right

$\hat{j}$ indicates up + down

  - 'i' and 'j' are the unit vectors in the x and y directions
  - x(t) and y(t) are real valued functions of the variable 't'
- Ex. (from Saxon Calc BC book):

Given $\vec{p}(t) = t^2\hat{i} + (3t - 2)\hat{j}$, find the velocity and acceleration functions:

1) To find $\vec{v}(t)$, differentiate the highlighted functions

$\vec{v}(t) = 2t\hat{i} + 3\hat{j}$

2) To find $\vec{a}(t)$, differentiate $\vec{v}(t)$ the same way

$\vec{a}(t) = 2\hat{i} + 0\hat{j} = 2\hat{i}$

- Integrating vector valued functions:
  - Similar to how you take the derivative of vector valued functions by differentiating x(t) and y(t), and leaving the 'i' and 'j', to find the integral

of vector-valued functions, integrate the real-valued functions and leave the ‘i’ and ‘j’

- Integrating a velocity vector gives you the displacement of the particle over the interval of time, and integrating a speed vector gives you the total distance travelled by the particle in that interval.
- Ex.:

$$\int (t^3\hat{i} - 2\hat{j})\,dt$$
$$= \int (t^3\,dt)\hat{i} - \int (2\,dt)\hat{j}$$
$$= \boxed{\left(\frac{t^4}{4} + C_1\right)\hat{i} - (2t + C_2)\hat{j}}$$

- **Polar curves:**
  - Polar coordinates: used to describe the location of points on a polar curve/graph
    - x = r$\cos\theta$ and y = r$\sin\theta$
      - *x and y are defined parametrically, in terms of ‘θ'
    - Differentiating in polar form:
      - To find $\frac{dy}{dx}$, use the equation for finding the slope of a parametric equation → $\frac{dy}{dx} = \frac{dy/d\theta}{dx/\,d\theta}$

Ex. Find the slope of the tangent line to the curve $r = 2 + 4\sin\theta$

plug the 'r' equation in

$x = r\cos\theta$
$x = (2+4\sin\theta)\cos\theta = 2\cos\theta + 4\sin\theta\cos\theta$

$y = r\sin\theta$
$y = (2+4\sin\theta)\sin\theta = 2\sin\theta + 4\sin^2\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos\theta + 8\sin\theta\cos\theta}{-2\sin\theta - 4\sin^2\theta + 4\cos^2\theta}$$

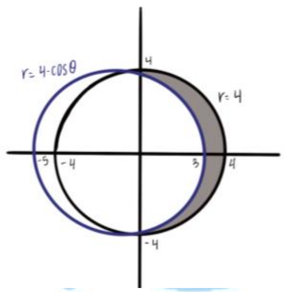
$$\frac{dy}{dx} = \frac{\cos\theta + 4\sin\theta\cos\theta}{2\cos^2\theta - 2\sin^2\theta - \sin\theta} = \boxed{\frac{\cos\theta + 2\sin 2\theta}{2\cos 2\theta - \sin\theta}}$$

.

  - **Finding the area of a single polar curve:**
    - A = $\int_a^b \frac{1}{2} r^2 d\theta$
    - Ex. Find the area of the region bounded by the polar curve r=1+$\cos\theta$

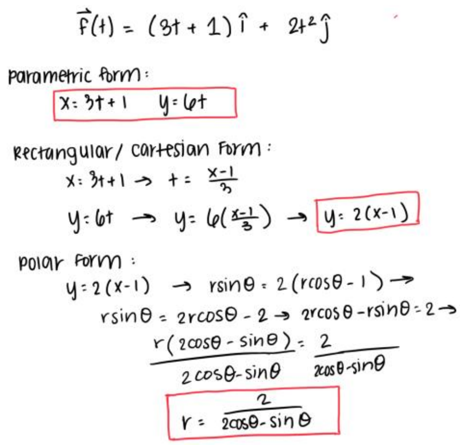
- $\frac{1}{2}\int_0^{2\pi} (1 + cos\theta)^2 d\theta$ → integrate that
    - *to find polar area, you will often need to know these trig identities:
        - $cos^2\theta = \frac{1}{2} + \frac{1}{2}cos\theta$
        - $sin^2\theta = \frac{1}{2} - \frac{1}{2}cos\theta$
    - To find the limits of the integral, find the θ points where r consists of that region.
        - In this case, it takes from 0 to 2π for "r" to be drawn completely, which is why those are the limits

- **Finding the area of a region bounded by 2 polar curves:**
    - A = $\int_a^b \frac{1}{2}(r_2^{\ 2} - r_1^{\ 2})\ d\theta$ with $r_2\ (\theta) > r_1(\theta)$
    - Ex. (from Princeton book): find the area inside the circle r = 4 and outside of the curve r = 4 - cosθ(the shaded region in the graph below)



- 1) To find the limits of the integral, set the two equations equal to each other, and solve for θ→ 4 = 4 – cosθ = cosθ $= 0$ at θ= -$\pi/2$ and $\pi/2$
    - You could also establish the limits of the integral as $\int_0^{\pi/2} r d\theta$, and multiply the whole integral by 2, since the graphs of the polar curves are symmetrical
- 2) find which function is on the "outside", and in this case, r = 4 is
    - $2\int_0^{\pi/2} [\frac{1}{2}(4)^2 - (4 - cos\theta)^2]\ d\theta$ OR $\int_{-\pi/2}^{\pi/2} [\frac{1}{2}(4)^2 - (4 - cos\theta)^2]\ d\theta$
- 3) integrate

- **Connecting polar, vector, and parametric functions:**
  - A function can be written in parametric, rectangular/Cartesian, or polar form.
  - Ex. (on the next page):



Unit 10: Infinite Sequences and Series

- **Difference between sequences and series:**
  - a_n = sequence
  - S = $\sum_{n=1}^{\infty} a_n$ = series
  - Sequences are lists of the terms, whereas series are lists of the addends of the series.
    - Series: to find the second term of the series, you ADD a_1 and a_2 together
      - This is known as partial sums → 2nd partial sum = $S_2 = a_1 + a_2$
        - nth partial sum = the sum of the first n terms of the series
- **Convergence/Divergence:**
  - A series converges only if the SEQUENCE OF PARTIAL SUMS (S_n) converges to a finite number. If it doesn't converge, then it diverges.
    - $\lim_{n\to\infty} S_n$ = finite / a real number
- **Geometric series:**
  - Defined as a series with a constant ratio between each term
  - In the form of $\sum_{n=1}^{\infty} ar^n$, with 'a' representing the 1st term of the geometric series, and 'r' being the ratio that's repeatedly expressed.

    - If $|r| < 1$, then the geometric series converges to $\frac{a}{1-r}$
        - $\lim_{n\to\infty} \sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$
        - Ex. $\lim_{n\to\infty} \sum_{n=1}^{\infty} (\frac{1}{2})^n \rightarrow$ r = $\frac{1}{2}$, which is < 1, so the series converges.
            - To find what it converges to, plug in numbers into the formula →
                - a = $(\frac{1}{2})^1$(a = the first term of the series, which means n=1)
                - R = $\frac{1}{2}$(r = the number that has an exponent ; the ratio)
                    - $\frac{a}{1-r} = \frac{(1/2)}{1-(1/2)} = \frac{(1/2)}{(1/2)} = 1 \rightarrow \lim_{n\to\infty} \sum_{n=1}^{\infty} (\frac{1}{2})^n = 1$
    - If $|r| > 1, or = 1,$then the series diverges.
        - $\lim_{n\to\infty} \sum_{n=1}^{\infty} ar^n =$ DNE
        - Ex. $\lim_{n\to\infty} \sum_{n=1}^{\infty} 2^n \rightarrow$ r = 2, which is > 1, so the series diverges.
    - Ex. determine if the series converges/diverges: $\sum_{n=1}^{\infty} \frac{3}{2^n}$
        - *it may seem like this series converges at first, since it seems like r > 1. However, be careful when determining the value of the ratio. In this example, it may seem like r = 3/2, but if you look at the equation, 3 isn't being raised to the nth power.
            - Thus, it helps to rearrange the equation → $\sum_{n=1}^{\infty} 3(\frac{1}{2})^n$
                - Now, it's easier to see that the series does converges and find what it converges to → $\lim_{n\to\infty} \sum_{n=1}^{\infty} 3(\frac{1}{2})^n = \frac{(3/2)}{1-(1/2)} =$ 3

- **The *n*th term test for divergence:**
    - Used to determine whether or not a function DIVERGES; it CANNOT tell you if a function converges
    - If $\lim_{n\to\infty} a_n \neq 0$, then $\lim_{n\to\infty} \sum_{n=1}^{\infty} a_n$diverges
        - Ex. $\sum_{n=1}^{\infty} n \rightarrow \lim_{n\to\infty} n = \infty \neq 0$
    - If $\lim_{n\to\infty} a_n = 0$, then the series' convergence cannot be determined yet.

- **Integral test for convergence:**
  - Improper integrals: any integral in the form of $\int_{c}^{\infty} f(x)\text{dx}$, $\int_{-\infty}^{\infty} f(x)\, dx$, or $\int_{-\infty}^{\square}$
    - If the limit of improper integrals exists, then the integral converges. If the limit DNE, then it diverges
    - Ex. $\int_{1}^{\infty} \frac{1}{x} dx = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{x} dx = \lim_{b\to\infty} \ln(\text{b}) - \ln(1) = \lim_{b\to\infty} \ln(\text{b}) = \infty \to$ integral DIVERGES because the limit of the integral isn't FINITE
- **Harmonic series and *p* series:**
  - *p* series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where *p* is a real number
    - Converges if p > 1
      - Ex. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because 3 > 1
      - *unlike geometric series, you can't determine what value a p-series converges to
    - Diverges if p ≤1
      - Ex. $\sum_{n=1}^{\infty} \frac{1}{n^{-3}}$ diverges because -3 < 1
  - **Harmonic series**: type of p series where p = 1 → $\sum_{n=1}^{\infty} \frac{1}{n}$
    - Diverges because p is equal to 1
    - Any multiple of the harmonic series also diverges
      - Ex. $\sum_{n=1}^{\infty} \frac{1}{2n}$ and $\sum_{n=1}^{\infty} \frac{5}{n}$ both diverge
    - However, alternating harmonic series ($\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$) do converge because as n → ∞,the series gets closer and closer to 0
  - **Comparison test for convergence:**
    - If $\sum_{n=1}^{\infty} a_n$ converges, and $c_n < a_n$, then $\sum_{n=1}^{\infty} c_n$ converges too.
      - Ex. determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ converges or diverges
        - $\frac{1}{n^2-1} < \frac{1}{n^2}$, and $\frac{1}{n^2}$ converges, so by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ also converges
    - If $\sum_{n=1}^{\infty} a_n$ diverges, and $c_n > a_n$, then $\sum_{n=1}^{\infty} c_n$ diverges too.

- **Limit comparison test:**

  If $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = c$, where $c =$ *finite, positive number*

  both $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converge or* **both** *diverge*

  - Ex. (from Rajendra Dahal's Limit Comparison test video)

    $$\sum_{n=1}^{\infty} \frac{1}{3n+5}$$

    Here, $a_n = \frac{1}{3n+5}$.

    Let $b_n = \frac{1}{n}$. Note that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges being a harmonic series.

    Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{3n+5} / \frac{1}{n}$

    $= \lim_{n\to\infty} \frac{1}{3n+5} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{3n+5}$

    - In the example above, the $\lim_{n\to\infty} \frac{n}{3n+5} = \frac{1}{3}$, which is a finite, positive number, and because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does the series $\sum_{n=1}^{\infty} \frac{1}{3n+5}$

- **Alternating series for convergence:**
  - This is used to test a series that alternates from negative↔positive
    - $\sum_{n=1}^{\infty} (-1)^n a_n$
  - **Alternating series test**: If you have an alternating series, and $\lim_{n\to\infty} |a_n| = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges
    - Ex. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$
      - $\lim_{n\to\infty} \left|(-1)^n \frac{1}{n}\right| = 0$, so $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges
- **Ratio test for convergence:**
  - Let r = $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ for $\sum_{n=1}^{\infty} a_n$
    - If r > 1, then the series diverges
    - If r < 1, then the series converges
    - If r = 1, then the convergence/divergence of the series is inconclusive

- Ex. determine if the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges or diverges

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2}{2n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{2}(1+\frac{1}{n})^2\right|$$

$$= \frac{1}{2} < 1$$

  - Because $r = \frac{1}{2} < 1$, the series converges

- **Absolute convergence:**
  - If an alternating series converges after taking the absolute value of the series function, then that series is said to converge absolutely
    - Basically, if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely
  - Ex. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \to \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^3}\right|$ converges because of p-series, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ is said to converge absolutely
- **Conditional convergence:**
  - If an alternating series converges, and does NOT converge absolutely, then it's said to converge conditionally
  - An alternating series that conditionally converges must follow these conditions:
    - a_n is positive for all n
    - $a_{n+1} \leq a_n$ for all n
    - $\lim_{n\to\infty} a_n = 0$
- **Alternating Series Error Bound:**
  - $|S - S_n| \leq |a_{n+1}|$
    - S = the actual sum of the series

- S_n =the partial sum of a convergent alternating series
- a_{n+1}= the first unused term

■ Ex. (from Princeton book): Find the error bound of $\sum_{n=1}^{7} \quad (-1)^{n+1}\frac{1}{n^2}$

- Take the absolute value of the first unused term
  - In this case, to find S_n, you added up the terms up until n=7, so the first unused term is when n=8
    - ■ $|S - S_7| \leq |a_{7+1}| \rightarrow (-1)^8 \frac{1}{8^2}$= 1/64 = error bound

- **Taylor polynomials:**
  - In the form of: $\sum_{n=0}^{\infty} \quad \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3...$
    - ■ Special case: when a=0, this is known as a Maclaurin series
    - ■ *f^njust represents the degree of the derivative
      - For ex., f^5stands for the fifth derivative of the function
  - Ex. Find the Taylor series of sinx expanded around x=π
    - ■ 1) find the derivatives of sinx, and plug in their values at x=π

n	f^n	$f^n(\pi)$
0	sin(x)	0
1	cos(x)	-1
2	-sin(x)	0
3	-cos(x)	1
4	sin(x)	0

- 2) plug in the values into the Taylor Series formula → Sinx = $\frac{0}{0!}(x-\pi)^0 - \frac{1}{1!}(x-\pi)^1 + \frac{0}{2!}(x-\pi)^2 + \frac{1}{3!}(x-\pi)^3 + .. =$ - (x -π) + $\frac{(x-\pi)^3}{3!}$- $\frac{(x-5)^5}{5!}$...

- 3) combine the terms into one series equation →

  $\sum_{n=0}^{\infty} \quad (-1)^{n+1} \frac{(x-\pi)^{2n+1}}{(2n+1)!}$

- Ex. Find the Taylor polynomial for e^xexpanded around x=0

n	$f^{(n)}$	$f^{(n)}(0)$
0	e^x	1
1	e^x	1
2	e^x	1
3	e^x	1
4	e^x	1
5	e^x	1

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \ldots$

↳ $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ = Taylor Polynomial for e^x

- *IMPORTANT - memorize the taylor polynomial for e^x!!

- **Approximations:**
  - To approximate the value of a function at a particular x value, just plug in the given x value into your equation
  - Ex. use the 4th degree Taylor polynomial to estimate e^3
    - Now that you know the Taylor polynomial for e^x,all you do to find $e^3$is to plug in x=3 into the first 4 terms of the series → e^3= 1 + 3 + $\frac{3^2}{2!} + \frac{3^3}{3!}$= 17.5

- **Lagrange Error Bound**
  - Lagrange's Remainder: $R_n = f^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!}$, where 'c' is in between 'x' and 'a'
    - $R_n$represents the remainder (error) AFTER the nth degree partial sum
  - Ex. approximate e with an error less than 0.0001
    - $\sum_{n=0}^{\infty} \quad \frac{x^n}{n!} \leq 0.001$
      - For this problem, you need to first find the value of 'n' that gives you a number that's ≤ 0.001

- The only way to do this is to plug in values
- For ex. When n=3, $a_3 = \frac{1^3}{3!} \approx 0.1667$
  - *you know that x=1 because the problem asks you to approximate e^1
  - 0.1667 > 0.001, so you have to keep on plugging in numbers for 'n', until you get a value ≤ 0.001
  - In this case, when n=8, $a_8 = \frac{1^8}{8!}$ ≈0.0000248, which is ≤ 0.001, and thus, to approximate e with an error less than 0.0001, find $\sum_{n=0}^{8} \frac{1^n}{n!}$

- **Radius and interval of convergence of power series:**
  - *remember: a power series: $\sum_{n=0}^{\infty} a_n(x-r)$
  - If a power series converges, it either converges over an interval, or to a single point.
    - Use the ratio test to determine interval of convergence:
      - Ex. (from Princeton) Find the interval of convergence for : $\sum_{n=0}^{\infty} \frac{x^n}{1+n^2}$

$$r = \lim_{n\to\infty} \left| \frac{x^{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x^n x\,(1+n^2)}{x^n(1+(n+1)^2)} \right| =$$

$$\lim_{n\to\infty} |x| \left(\frac{1+n^2}{n^2+2n+2} \right) = |x| \lim_{n\to\infty} \left(\frac{1+n^2}{n^2+2n+2} \right) =$$

$$|x| < 1 \rightarrow -1 < x < 1$$

↑ interval of convergence

* Make sure you check to see if the function also converges @ the end points!!

$$x=1 \rightarrow \sum_{n=0}^{\infty} \frac{x^n}{1+n^2} = \sum_{n=0}^{\infty} \frac{1}{1+n^2} = \text{converges}$$

$$x=-1 \rightarrow \sum_{n=0}^{\infty} \frac{x^n}{1+n^2} = \sum_{n=0}^{\infty} -\frac{1}{1+n^2} = \text{converges}$$

interval of convergence: $[-1, 1]$

- You have to check the endpoints of the interval to see if the function converges at the endpoints! If it does converge at an endpoint, place the interval in brackets, and if not, place the interval in parentheses.

- If it converges at 1 endpoint and not the other, denote it like this: [a,b) or (a,b], with the bracket indicating convergence at that endpoint

- **Radius of convergence:**
  - Similar to how the radius of a circle describes the length from the outer edge of the circle to the centerpoint, the radius of convergence describes the distance between the midpoint and the endpoint of the interval of convergence.
  - In the example above, the interval of convergence was [-1,1]. Thus, the radius of convergence is 1, since the midpoint of that interval is 0, and the distance between 0 (the midpoint) and 1 (the endpoint) is 1.

- **Representing functions as Power series:**
  - Differentiation of power series
    - 1) write out the first few terms of the power series
      - Ex. f(x) = e^x= 1 + x + $\frac{x^2}{2!}$+ $\frac{x^3}{3!}$+ $\frac{x^4}{4!}$ +...
    - 2) differentiate the written terms as you normally would
      - Ex. f'(x) = e^x= 1 +x +$\frac{x^2}{2!}$+ $\frac{x^3}{3!}$+ ....
    - 3) rewrite the new Taylor polynomial (if needed)
      - Ex. f'(x) = $\sum_{n=0}^{\infty} \frac{(x)^{n+1}}{(n+1)!}$
  - **Integration of power series:**
    - Same steps as differentiating power series goes for integration, except for you're going to integrate the function instead of differentiating it
      - Ex. h(x) = 1 + x + $x^2 + x^3 + x^4 +$... (Maclaurin series for -ln(1-x)). If g'(x) = h(x), then find the power series for g(x)
        - $\int (1 + x + x^2 + x^3 + x^4 + \ldots)$dx = x + $\frac{x^2}{2}$+ $\frac{x^3}{3}$+ $\frac{x^4}{4}$ +...

          =$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$
  - **Substitution of power series:**

- If f(u) = $\sum_{n=0}^{\infty} a_n u^n$, and u = g(x), then f(u) = f(g(x)) = $\sum_{n=0}^{\infty} a_n (g(x))^n$
    - Ex. the power series for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

      The power series for $e^{x^3} = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$
        - Notice how to find the power series for e^{x^3}, you just plug in (x^3) into the power series for the original function, e^x

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