

AS Further Mathematics 8FM0

Specimen Paper - Further Pure Mathematics 2 Mark Scheme

| Question | Scheme | Marks | AOs | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|--|--|------------|--------------|----|----|----|----|---|---|---|---|---|----|----|---|---|---|----|---|---|----|---|---|----|---|----|---|---|---|---|---|----|---|----|---|----|----|---|---|----|---|---|----|----|----|---|---|---|---|---|----|------|
| 1(i)(a) | \times_{13} <table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td>1</td><td>3</td><td>4</td><td>9</td><td>10</td><td>12</td></tr> <tr><td>1</td><td>1</td><td>3</td><td>4</td><td>9</td><td>10</td><td>12</td></tr> <tr><td>3</td><td>3</td><td>9</td><td>12</td><td>1</td><td>4</td><td>10</td></tr> <tr><td>4</td><td>4</td><td>12</td><td>3</td><td>10</td><td>1</td><td>9</td></tr> <tr><td>9</td><td>9</td><td>1</td><td>10</td><td>3</td><td>12</td><td>4</td></tr> <tr><td>10</td><td>10</td><td>4</td><td>1</td><td>12</td><td>9</td><td>3</td></tr> <tr><td>12</td><td>12</td><td>10</td><td>9</td><td>4</td><td>3</td><td>1</td></tr> </table> | 1 | 3 | 4 | 9 | 10 | 12 | 1 | 1 | 3 | 4 | 9 | 10 | 12 | 3 | 3 | 9 | 12 | 1 | 4 | 10 | 4 | 4 | 12 | 3 | 10 | 1 | 9 | 9 | 9 | 1 | 10 | 3 | 12 | 4 | 10 | 10 | 4 | 1 | 12 | 9 | 3 | 12 | 12 | 10 | 9 | 4 | 3 | 1 | One non-identity row or columns correct | M1 | 1.1b |
| | 1 | 3 | 4 | 9 | 10 | 12 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | 1 | 1 | 3 | 4 | 9 | 10 | 12 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | 3 | 3 | 9 | 12 | 1 | 4 | 10 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 4 | 12 | 3 | 10 | 1 | 9 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 9 | 9 | 1 | 10 | 3 | 12 | 4 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 10 | 10 | 4 | 1 | 12 | 9 | 3 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 12 | 12 | 10 | 9 | 4 | 3 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| At least four rows or four columns correct | A1 | 1.1b | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| All correct | A1 | 1.1b | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | (3) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (i)(b) | {1,3,9} (This is the only such subgroup) | B1 | 1.1b | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | | (1) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (i)(c) | The order of the group is 6, and 4 does not divide 6, so there can be no subgroup of order 4 by Lagrange's Theorem. | B1 | 2.4 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | | (1) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (ii) | Multiplicative identity is 1, but e.g. $1 \times 3 \equiv_{15} 3$, $3 \times 3 \equiv_{15} 9$, $6 \times 3 \equiv_{15} 3$, $9 \times 3 \equiv_{15} 12$ and $12 \times 3 \equiv_{15} 6$ | M1 A1 | 1.1b 1.1b | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | So 3 has <u>no (left) inverse</u> element and hence $(\{1,3,6,9,12\}, \times_{15})$ is <u>not a group</u> . | A1 | 2.3 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | | (3) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (8 marks) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Notes: | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| <p>(i)(a) M1: At least one non-identity row or column must be correct A1: Allow for four correct rows or four correct columns (including a border). A1: Fully correct.</p> <p>(i)(b) B1: Correct subgroup found.</p> <p>(i)(c) B1: States both that the order of the group is 6 and 4 does not divide 6, and refers to Lagrange's Theorem (either by name or by quoting the result "the order of a subgroup divides the order of the group" or similar).</p> <p>(ii) M1: Identifies 1 as the identity and shows that there is an element with no inverse. E.g. may write out a partial group table and state no 1 in second row. Must give evidence -- just 'no inverses' is M0. A1: Correct multiples of 3 (all checked) or their value or correct row in table if drawn. A1: Concludes <u>not a group</u> with reason <u>inverse axiom fails</u>. Accept 'no inverses' as reason providing evidence has been given to award the M.</p> | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

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|----------|---|-------|------|
| 2(a) | $x_2 = 5 \times 2 - 1 = 9$ and $y_2 = 3 \times 2 + 1 = 7$, so $x_3 = 5 \times 9 - 7$ | M1 | 11b |
| | Hence $x_3 = 38$ | A1 | 1.1b |
| | | (2) | |
| (b) | (i) $\begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(1-\lambda) - 3 \times -1 = 0$ $\Rightarrow \lambda^2 - 6\lambda + 8 = 0 \Rightarrow ((\lambda - 2)(\lambda - 4) = 0 \Rightarrow) \lambda = ..$ | M1 | 1.1b |
| | $\lambda = 2$ and $\lambda = 4$ | A1 | 1.1b |
| | (ii) $\lambda = 2 \Rightarrow \begin{cases} 5x - y = 2x \\ 3x + y = 2y \end{cases} \Rightarrow 3x = y$ oe or $\lambda = 4 \Rightarrow \begin{cases} 5x - y = 4x \\ 3x + y = 4y \end{cases} \Rightarrow x = y$ oe | M1 | 1.1b |
| | Either $\lambda = 2 \Rightarrow \mathbf{v} = k \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ or $\lambda = 4 \Rightarrow \mathbf{v} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | A1 | 1.1b |
| | Both $\lambda = 2 \Rightarrow \mathbf{v} = k \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\lambda = 4 \Rightarrow \mathbf{v} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | A1 | 1.1b |
| | | (5) | |
| (c) | $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ ft their eigenvalues either way round, OR $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$ ft their eigenvectors as columns in any order. | B1ft | 1.1b |
| | Deduces BOTH $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ AND $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$ ft their eigenvalues either way round and their eigenvectors (any multiples) as columns <i>in the correct order corresponding to their eigenvalues</i> . | B1ft | 2.2a |
| | | (2) | |
| (d) | $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{P}^{-1} \mathbf{A} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ OR $\mathbf{D} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \mathbf{P}^{-1} \mathbf{A} \mathbf{P} \mathbf{P}^{-1} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ | M1 | 1.1b |
| | $\dots = \mathbf{P}^{-1} \mathbf{A} \mathbf{P} \mathbf{P}^{-1} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{D} \begin{pmatrix} u_n \\ v_n \end{pmatrix} *$ OR $\dots = \mathbf{P}^{-1} \mathbf{A} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} *$ | A1* | 2.1 |
| | | (2) | |

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| (e) | $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{D} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \Rightarrow u_{n+1} = 2u_n \text{ and } v_{n+1} = 4v_n$ $\Rightarrow u_n = u_1 \times 2^{n-1 \text{ or } n} \text{ and } v_n = v_1 \times 4^{n-1 \text{ or } n}$ | M1 | 3.1a |
| | $u_n = u_1 \times 2^{n-1} \text{ and } v_n = v_1 \times 4^{n-1} \text{ f.t. their eigenvalues in place of 2 and 4.}$ | A1ft | 1.1b |
| | $\text{So } \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{P} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_n + v_n \\ 3u_n + v_n \end{pmatrix} = \begin{pmatrix} u_1 \times 2^{n-1} + v_1 \times 4^{n-1} \\ 3u_1 \times 2^{n-1} + v_1 \times 4^{n-1} \end{pmatrix}$ | M1 | 2.1 |
| | $\text{OR } \begin{cases} u_n = -\frac{1}{2}x_n + \frac{1}{2}y_n \\ v_n = \frac{3}{2}x_n - \frac{1}{2}y_n \end{cases} \Rightarrow \begin{cases} x_n = u_n + v_n = u_1 \times 2^{n-1} + v_1 \times 4^{n-1} \\ y_n = 3u_n + v_n = 3u_1 \times 2^{n-1} + v_1 \times 4^{n-1} \end{cases}$ | | |
| | $x_1 = 2, y_1 = 1 \Rightarrow u_1 = -\frac{1}{2} \text{ and } v_1 = \frac{5}{2} \text{ found using matrix multiplication or equations.}$ | M1 | 1.1b |
| | $\text{So } x_n = \left(-\frac{1}{2}\right)2^{n-1} + \left(\frac{5}{2}\right)4^{n-1} \text{ and } y_n = \left(-\frac{3}{2}\right)2^{n-1} + \left(\frac{5}{2}\right)4^{n-1} \text{ oe}$ | A1 | 1.1b |
| | | (5) | |
| (16 marks) | | | |
| Notes: | | | |
| (a) | M1: Finds both x_2 and y_2 and attempts x_3 using them. A1: Correct $x_3 = 38$ | | |
| (b)(i) | M1: Forms the characteristic equation for A and attempts to solve (usual rules). A1: Correct eigenvalues. | | |
| (b)(ii) | M1: Correct equations set up for at least one eigenvalue and attempt to solve. A1: One correct eigenvector. A1: Both correct. | | |
| (c) | B1ft: Either gives D as the diagonal matrix with their eigenvalues (either way) on diagonal, or P as the matrix with their eigenvectors (any multiples) as columns. B1ft: Both P as the matrix with their eigenvectors (any multiples) as the columns AND D as the diagonal matrix with the corresponding eigenvalues on diagonal <u>in the correct order to match the columns of P</u> . | | |

Question 2 notes continued

(d)

M1: Changes from $\{u, v\}$ system to $\{x, y\}$ system and applies the recurrence relation to $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$. If

working in reverse, for this mark both **D** and $\begin{pmatrix} u_n \\ v_n \end{pmatrix}$ must be replaced.

A1*: Correctly derives result by introducing $\mathbf{P}\mathbf{P}^{-1}$ before applying $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ and using the transformation again. If working in reverse it is for completing the proof correctly.

(e)

M1: Uses their diagonal matrix to form two separate first order recurrence relations in one variable and attempts to solve each. Allow if the index is n instead of $n-1$.

A1ft: Correct form for each, $u_1 \times \lambda_1^{n-1}$ and $v_1 \times \lambda_2^{n-1}$ for their eigenvalues. May find the closed forms for u_n and v_n using $u_1 = -\frac{1}{2}$ and $v_1 = \frac{5}{2}$ here, but the A1 needs only the correct structure.

M1: Uses the matrix equation or solves simultaneous equations in the x_n and y_n sequences, with closed forms for u_n and v_n , to obtain forms for x_n and y_n .

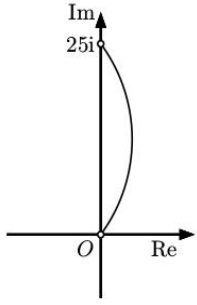
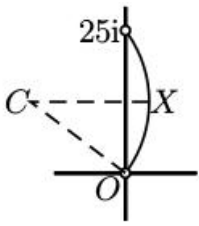
M1: Uses initial terms to find the constants -- may be awarded before the previous method mark.

A1: Correct forms for both sequences, need not be simplified, but accept any equivalents such as

$$x_n = 5 \times 2^{2n-3} - 2^{n-2} = \frac{1}{8} 2^n (5 \times 2^n - 2) \text{ and } y_n = 5 \times 2^{2n-3} - 3 \times 2^{n-2} = \frac{1}{8} 2^n (5 \times 2^n - 6)$$

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| 3(a) |  <p>M1 - any arc from O to $25i$ in the first quadrant OR a minor arc from O to $\pm 25i$ (in any quadrant) A1 - minor arc from O to $25i$ in first quadrant.</p> | M1 A1 | 1.1b 1.1b |
| | | (2) | |
| (b) | $y = \frac{25}{2}i$ where centre is $x + yi$ | B1 | 2.2a |
| <p>Circle passes through O so $r^2 = \left(-\frac{25}{2}(2+\sqrt{3})\right)^2 + \left(\frac{25}{2}\right)^2$ \newline OR passes through $25i$ so $r^2 = \left(-\frac{25}{2}(2+\sqrt{3})\right)^2 + \left(25 - \frac{25}{2}\right)^2$</p>  <p>ALT uses circle geometry to deduce angle OCX is $\frac{\pi}{12}$ or angle CXO is $11\pi/24$ and attempts to use in a correct triangle.</p> | | M1 | 3.1a |
| $\Rightarrow r = \sqrt{25^2 \left(\frac{1}{4}(4+4\sqrt{3}+3) + \frac{1}{4}\right)} = 25\sqrt{(2+\sqrt{3})}$ ALT $r = \frac{12.5}{\sin\left(\frac{\pi}{12}\right)}$ or $r = \frac{25}{2}(2+\sqrt{3}) + \frac{25/2}{\tan(11\pi/24)}$ oe | | M1 | 1.1b |
| $\Rightarrow r = \frac{25}{2}(\sqrt{6} + \sqrt{2})$ or $r = \text{awrt}48.296$ ($r = 48.29629131\dots$) | | A1 | 1.1b |
| | | (4) | |
| (c) | <p>Distance from centre of arc to first ball is $\sqrt{\left(0.5 - \left(-\frac{25}{2}(2+\sqrt{3})\right)\right)^2 + (23.25 - 12.5)^2}$</p> <p>$\left(= \sqrt{47.150\dots^2 + 10.75^2} = \sqrt{2338.7448\dots}\right) = 48.3605\dots$ (awrt 48.36)</p> <p>So difference between distance to ball and radius of arc is $48.3605\dots - 48.2962\dots = 0.0642\dots$</p> <p>$0.064 < 0.1$ (=twice radius) and so the balls will collide</p> | M1 A1 M1 A1 | 3.1b 3.4 1.1b 3.2a |
| | | (4) | |
| (10 marks) | | | |

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Question 3 notes:

(a)

M1: See scheme.

A1: A correct sketch as per scheme.

(b)

B1: Deduces the imaginary part of the centre is $\frac{25}{2}i$.

M1: Uses that the circle passes through O or $25i$ with Pythagoras to form an equation for r^2 with x and their y . Alternatively, may use circle geometry to find the one of the angles shown with attempt to use it.

M1: Proceeds to find r .

A1: Correct radius, either exact or awrt to 3 d.p.

(c)

M1: Realises the need to find the distance of the first ball from the centre of the circle, so applies Pythagoras to find this.

A1: Correct distance.

M1: Compares distance to radius of circle.

A1: Correct difference, accept (\pm) awrt 0.06, compared to twice the radius and conclusion that second ball will hit the first.

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|----------|--|-------|------|
| 4 | First fact gives $a - b + c = 11k$, $k \in \mathbb{Z}$ OR second fact gives $a + b + c = 2m + 1$, $m \in \mathbb{Z}$ OR third fact gives $(100a + 10b + c \equiv 5 \pmod{9} \Rightarrow) a + b + c \equiv 5 \pmod{9}$ or $a + b + c = 5 + 9p$, some $p \in \mathbb{Z}$ | M1 | 1.1b |
| | Facts 1 and 2 give $a - b + c = 2(m - b) + 1$ is odd (oe). Facts 1 and 3 give $2b = 5 + 9p - 11k$ or $2(a + c) = 5 + 11k + 9p$ where $k \in \{0, 1\}$, $p \in \{0, 1, 2\}$ Facts 2 and 3 give $a + b + c \equiv 5 + 18q$, $q \in \{0, 1\}$ | M1 | 3.1a |
| | Facts 1 and 2: must reach $a - b + c = 11$ Facts 1 and 3: must reach $2b = 12$ or 14 (both values needed) or $2(a + c) = 14$ or 16 or 34 (all three needed). Facts 2 and 3: must reach $a + b + c = 5$ or 23 (both values needed) | A1 | 2.2a |
| | Uses the remaining fact to find a value for b (or for $a + c$ using facts 1 and 3 initially). | M1 | 3.1a |
| | For $b = 6$ or for $a + c = 17$ | A1 | 2.2a |
| | & Thus as $a + c = 17$ so a and c must be 8 and 9, so the possibilities are 968 or 869 and no others. | A1 | 2.1 |
| | | (6) | |
| | | | |

(6 marks)

Notes:

M1: Uses any one of the facts to form an equation in a , b and c , e.g. property of divisibility by 11, to form equation $a - b + c = 11k$ ($k \in \mathbb{Z}$) (or $k \in \{0, 1\}$ if range of values considered here), OR second fact gives $a + b + c = 2k + 1$ (don't accept just $a + b + c$ is odd for this mark), OR achieves $a + b + c \equiv 5 \pmod{9}$ oe from third fact (must reduce the coefficients modulo 9).

M1: Uses a second property AND combines two properties together. E.g. fact 2 gives $a + b + c = 2m + 1$ ($m \in \mathbb{Z}$) oe, and combining with fact one we have $a - b + c = a + b + c - 2b$ is also odd, (or adds equations to deduce $2b = 2m + 11k + 1 \Rightarrow k$ odd).
Alternatives include e.g. $a + b + c - 5 \in \{0, 9, 18\}$ from fact three (since these are the only multiples of 9 possible for the ranges on a , b and c) and so $a + b + c - 5 = 0$ or 18 (allow the M if the 0 case is missing).

A1: Correctly combines two facts with consideration of the range of values to deduce a correct equation in just a , b and c . Possibilities are $a - b + c = 11$ or $a + b + c = 5$ or 23 oe. (Should have both values in the latter case unless a convincing reason why it is not 5 has been given.)

M1: Uses the remaining property and attempts to solve for b . E.g. third fact gives $100a + 10b + c \equiv 5 \pmod{9}$ and reduces modulo 9 to $a + b + c \equiv 5 \pmod{9}$ and combines with $a - b + c = 11$ to deduce $2b = 9p - 6$ oe so $b = \dots$

Use of facts 1 and 3 initially will require some work of trial and elimination of possibilities here. The cases that cannot happen must be seen to be rejected.

A1: Correct deduction that $b = 6$.

A1: Completes the argument and deduces correctly that the only two possible values are 968 and 869.

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