Q	Scheme	Marks	AOs	Pearson Progression Step and Progress Descriptor
1a	$\frac{dS}{dt} = 10\sinh 2t - 6\cosh 2t$	B1	1.1b	4th Be able to differentiate standard hyperbolic functions
	$10\sinh 2t - 6\cosh 2t = 0 \Rightarrow \tanh 2t = \frac{3}{5}$	M1	3.1b	
	$t = \frac{1}{2}\text{artanh}\,\frac{3}{5}$	A1	1.1b	
	$t = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\ln\left(\frac{1+\frac{3}{5}}{1-\frac{3}{5}}\right) = \frac{1}{4}\ln 4 = \ln\sqrt{2}$	M1	1.1b	
Alt	$S = \frac{5\left(e^{2t}+e^{-2t}\right)}{2} - \frac{3\left(e^{2t}-e^{-2t}\right)}{2} = e^{2t} + 4e^{-2t}$	M1	1.1b	
	$\frac{dS}{dt} = 2e^{2t} - 8e^{-2t}$	B1	1.1b	
	$2e^{2t} - 8e^{-2t} = 0 \Rightarrow 2e^{4t} = 8 \Rightarrow t = \frac{1}{4}\ln 4$	M1 A1	3.1b 1.1b	
	$S = 5\cosh(2\ln\sqrt{2}) - 3\sinh(2\ln\sqrt{2})$ $= \frac{5\left(e^{\ln 2}+e^{-\ln 2}\right)}{2} - \frac{3\left(e^{\ln 2}-e^{-\ln 2}\right)}{2}$	M1	1.1b	
	$= \frac{5\left(2+\frac{1}{2}\right)}{2} - \frac{3\left(2-\frac{1}{2}\right)}{2} = \frac{25}{4} - \frac{9}{4} = 4$ *	A1*	2.1	
		(6)		
1b	$\frac{d^2S}{dt^2} = 20\cosh 2t - 12\sinh 2t$ or $4e^{2t} + 16e^{-2t}$	B1ft	3.1b	4th Be able to differentiate standard hyperbolic functions
	$t = \ln\sqrt{2} \Rightarrow \frac{d^2S}{dt^2} = 20\left(\frac{5}{4}\right) - 12\left(\frac{3}{4}\right)$ or $4(2) + 16\left(\frac{1}{2}\right)$ $= 16, > 0, \therefore$ minimum	B1cao	2.4	
		(2)		
				(8 marks)

Notes

1a **B1:** Correct differentiation

M1: Sets derivative = 0 and obtains tanh $2t = k,\ k \neq 0$

A1: Correct value for t

M1: Uses correct logarithmic form of artanh x

Alt **M1:** Uses correct exponential forms of sinh $2t$ and cosh $2t$

B1: Correct differentiation

M1: Sets derivative = 0 and reaches $t = ...$

A1: Correct value for t

M1: Uses correct exponential forms of sinh $2t$ and cosh $2t$

A1*: Achieves given answer with no errors

1b **B1ft:** Correct differentiation of their derivative $p \sinh 2t + q \cosh 2t$ or $pe^{2t} + qe^{-2t},\ p,\ q \neq 0$

B1: Obtains correct $\frac{d^2S}{dt^2}$, shows it is positive and concludes minimum. If no substitution of $t = \ln\sqrt{2}$ is seen there must be justification of $\frac{d^2S}{dt^2} > 0$ (e.g. an appropriate sketch graph of $y = 20\cosh 2t - 12\sinh 2t$)

Q	Scheme	Marks	AOs	Pearson Progression Step and Progress Descriptor
2a	$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$	M1	1.1b	3rd Understand the definitions of hyperbolic functions
	$= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4}$	M1	1.1b	
	$= \frac{4}{4} = 1^*$	A1*	2.1	
Alt	$\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x)$	M1	1.1b	
	$= \left(\frac{e^x + e^{-x} + e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x} - e^x + e^{-x}}{2}\right)$	M1	1.1b	
	$= \frac{(2e^x)(2e^{-x})}{4} = \frac{4}{4} = 1^*$	A1*	2.1	
		(3)		
2b	$\frac{2\sinh x}{\cosh x} = \cosh x$	M1	1.1b	3rd Understand the definitions of hyperbolic functions
	$2 \sinh x = \cosh^2 x$ $2 \sinh x = 1 + \sinh^2 x$ $\sinh^2 x - 2 \sinh x + 1 = 0$	M1	1.1b	
	$(\sinh x - 1)^2 = 0$ $\sinh x = 1$	M1 A1	1.1b 1.1b	
	$x = \text{arsinh}\,1 = \ln\left(1 + \sqrt{1^2 + 1}\right) = \ln\left(1 + \sqrt{2}\right)$	A1	1.1b	
		(5)		
				(8 marks)

Notes

2a **M1:** Uses correct exponential forms of sinh x and cosh x

M1: Expands at least one bracket correctly

A1*: Obtains 1 with no errors. Both LHS and RHS of given answer to have been seen

Alt **M1:** Correct factorisation of $\cosh^2 x - \sinh^2 x$

M1: Uses correct exponential forms of sinh x and cosh x

A1*: Obtains 1 with no errors. Both LHS and RHS of given answer to have been seen

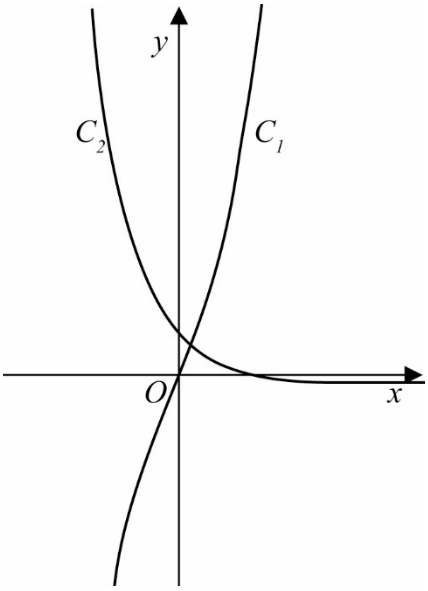
2b **M1:** Uses $\tanh x = \frac{\sinh x}{\cosh x}$

M1: Uses $\cosh^2 x = 1 + \sinh^2 x$ and obtains 3TQ in sinh x

M1: Solves their 3TQ in sinh x (usual rules)

A1: Correct value for sinh x

A1: $\ln\left(1+\sqrt{2}\right)$ only

Q	Scheme	Marks	AOs	Pearson Progression Step and Progress Descriptor
3a		**B1** **B1**	1.1b 1.1b	3rd Sketch the graphs of hyperbolic functions
	$y = -1$	**B1**	2.2a	
	(0, 5)	**B1**	1.1b	
	(5 ln 6, 0)	**B1**	1.1b	
		(5)		

3b	$12\sinh\frac{x}{5} = 6e^{-\frac{x}{5}} - 1$ $12\left(\frac{e^{\frac{x}{5}} - e^{-\frac{x}{5}}}{2}\right) = 6e^{-\frac{x}{5}} - 1$	**M1**	3.1a	3rd Understand the definitions of hyperbolic functions
	$6e^{\frac{x}{5}} - 6e^{-\frac{x}{5}} = 6e^{-\frac{x}{5}} - 1$ $6e^{\frac{x}{5}} - 12e^{-\frac{x}{5}} + 1 = 0$ $6\left(e^{\frac{x}{5}}\right)^2 + e^{\frac{x}{5}} - 12 = 0$	**M1**	1.1b	
	$\left(3e^{\frac{x}{5}} - 4\right)\left(2e^{\frac{x}{5}} + 3\right) = 0$ $e^{\frac{x}{5}} = \frac{4}{3}$	**M1** **A1**	1.1b 1.1b	
	$x = 5\ln\frac{4}{3}$	**A1cao**	1.1b	
		(5)		
				(10 marks)

Notes

3a **B1:** Uses the correct logarithmic form of artanh x with x =

B1: Correct **shape** of C_2

B1: $y = -1$ only (asymptote must be seen as an equation). Withhold if other asymptotes are given

B1: (0,5) - allow 5 indicated on y-axis

B1: (5 ln 6, 0) or exact equivalent (e.g. $-5\ln\frac{1}{6}$). Allow if indicated on x-axis

If only coordinates are seen and answers are given as (5, 0) and (0, 5 ln 6) score B1 B0

3b **M1:** Substitutes correct exponential form of $\sinh\frac{x}{5}$ into equation of intersection

M1: Obtains 3TQ in $e^{\frac{x}{5}}$

M1: Solves their 3TQ in $e^{\frac{x}{5}}$ (usual rules)

A1: Obtains $e^{\frac{x}{5}} = \frac{4}{3}$

A1cao: $x = 5\ln\frac{4}{3}$ only. Withhold if any incorrect work is seen (e.g. a second answer)

Q	Scheme	Marks	AOs	Pearson Progression Step and Progress Descriptor
4a	$\text{artanh}\left(\frac{12}{13}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{12}{13}}{1-\frac{12}{13}}\right)$	M1	1.1b	4th Understand and use the domains and ranges of inverse hyperbolic functions
	$= \frac{1}{2}\ln 25$ or $\ln 5$ $\Rightarrow k = 5$*	A1	2.1	
		(2)		
4b	$y = x \text{ artanh } 3x \quad u = x \quad v = \text{artanh } 3x$ $\frac{dv}{dx} = \frac{3x}{1-(3x)^2}$	B1	1.1b	5th Be able to differentiate hyperbolic functions using chain, product and quotient rules
	$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = 1(\text{artanh}\,3x) + x\frac{3x}{1-(3x)^2}$	M1	1.1b	
	$\frac{d}{dx}(x\text{ artanh }3x) = \text{artanh }3x + \frac{3x}{1-9x^2}$	A1	1.1b	
		(3)		
4c	$\int \text{artanh }3x\,dx = x\text{ artanh }3x - \int \frac{3x}{1-9x^2}dx$	M1	2.2a	7th Be able to solve calculus problems with hyperbolic functions in a range of familiar contexts
	$\int \frac{3x}{1-9x^2}dx = -\frac{1}{6}\ln(1-9x^2)$	M1 A1	1.1b 1.1b	
	$I = \int_0^{\frac{1}{9}} \text{artanh }3x\,dx = \left[x\,\text{artanh}\,3x + \frac{1}{6}\ln(1-9x^2)\right]_0^{\frac{1}{9}}$ $= \frac{1}{9}\text{artanh}\left(3\left(\frac{1}{9}\right)\right) + \frac{1}{6}\ln\left(1-9\left(\frac{1}{9}\right)^2\right) - \left(0 + \frac{1}{6}\ln 1\right)$	dM1	1.1b	
	$\text{artanh}\,\frac{1}{3} = \frac{1}{2}\ln\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}\ln 2$	M1	1.1b	

	$I = \frac{1}{18}\ln 2 + \frac{1}{6}\ln\frac{8}{9} = \frac{1}{18}\left(\ln 2 + 3\ln\frac{8}{9}\right) = \frac{1}{18}\left(\ln 2 + \ln\frac{512}{729}\right)$ $= \frac{1}{18}\ln\frac{1024}{729}$	**A1**	1.1b	
		(6)		
(11 marks)				

Notes

4a **M1:** Uses the correct logarithmic form of artabh x with $x = \frac{12}{13}$

A1*: Obtains $k = 5$ with an intermediate step and no errors seen

4b **B1:** $\frac{\mathrm{d}}{\mathrm{d}x}(\text{artanh } 3x) = \frac{3x}{1-(3x)^2}$

M1: Uses correct product rule

A1: Correct derivative

4c **M1:** Rearranges the result from part **b** or uses integration by parts in the correct direction

M1: Integrates $\frac{px}{1+qx^2}$ to obtain $r\ln(1 + qx^2)$, p, q, $r \neq 0$

A1: Correct derivative

dM1: Substitutes correct limit. The substitution of the limit 0 must be seen and subtracted if it does not give 0. Dependent on first two method marks

M1: Uses correct logarithmic form of artanh x

A1: $\frac{1}{18}\ln\frac{1024}{729}$ only

Q	Scheme	Marks	AOs	Pearson Progression Step and Progress Descriptor
5a	$\frac{\mathrm{d}x}{\mathrm{d}u} = 3\sinh u$	B1	1.1b	6th Be able to integrate functions using hyperbolic substitutions
	$I = \sqrt{x^2 - 9}\ \mathrm{d}x = \int \sqrt{9\cosh^2 u - 9}\ .\ 3\sinh u\ \mathrm{d}u$	M1	1.1b	
	$= 9\int \sqrt{\cosh^2 u - 1}\ .\ \sinh u\ \mathrm{d}u = 9\int \sinh^2 u\ \mathrm{d}u$	M1	1.1b	
	$= \frac{9}{2}\int (\cosh 2u - 1)\ \mathrm{d}u$	M1	1.1b	
	$= \frac{9}{2}\left(\frac{1}{2}\sinh 2u - u\right)$	A1	1.1b	
	$= \frac{9}{2}(\sinh u \cosh u - u) = \frac{9}{2}\left(\cosh u\sqrt{\cosh^2 u - 1} - u\right)$	M1	2.2a	
	$\cosh u = \frac{x}{3},\ u = \operatorname{arcosh}\frac{x}{3} \Rightarrow I = \frac{9}{2}\left(\frac{x}{3}\sqrt{\left(\frac{x}{3}\right)^2 - 1} - \operatorname{arcosh}\frac{x}{3}\right)$	M1	1.1b	
	$I = \frac{x}{2}\sqrt{x^2 - 9} - \frac{9}{2}\operatorname{arcosh}\frac{x}{3} + c$ *	A1*	2.1	
		(8)		

5bi	For $t = 2$ ($t < 3$), $v = \sqrt{t^2 - 9}$ is undefined since $t^2 - 9 (= -5) < 0$	**B1**	3.5b	8th Be able to solve calculus problems with hyperbolic functions in a range of unfamiliar contexts
		(1)		
5bii	distance to owner $= 15 - \int_3^5 \sqrt{t^2 - 9}\ \mathrm{d}t$	**M1**	**3.1**	
	$\int_3^5 \sqrt{t^2 - 9}\ \mathrm{d}t = \frac{5}{2}\sqrt{16} - \frac{9}{2}\operatorname{arcosh}\frac{5}{3} - \left(0 - \frac{9}{2}\operatorname{arcosh} 1\right)$	**M1**	**1.1b**	
	$\operatorname{arcosh}\frac{5}{3} = \ln\left(\frac{5}{3} + \sqrt{\left(\frac{5}{3}\right)^2 - 1}\right) = \ln\left(\frac{5}{3} + \frac{4}{3}\right)$	**M1**	**1.1b**	
	distance to owner $= 15 - \left(10 - \frac{9}{2}\ln 3\right) = 5 + \frac{9}{2}\ln 3$	**A1**	**3.4**	
		(4)		
				(13 marks)

Notes

5a **B1:** Correct derivative

M1: Uses their $\frac{\mathrm{d}x}{\mathrm{d}u}$ and $x = 3\cosh u$ to obtain a complete substitution from x to u

M1: Uses $\sinh^2 u = \pm\cosh^2 u \pm 1$

M1: Uses $\sinh^2 u = \pm\frac{1}{2}\cosh 2u \pm \frac{1}{2}$

A1: Correct integration

M1: Uses the correct hyperbolic identities $\sinh 2u = \sinh u \cosh u$ and $\sinh u = \sqrt{\cosh^2 u - 1}$ to obtain an expression in u and $\cosh u$ only

M1: Uses $\cosh u = \frac{x}{3}$ and $u = \operatorname{arcosh}\frac{x}{3}$ to obtain an expression in x only

A1*: Obtains given answer with no errors

5bi **B1:** Accept equivalents but answer must explain why the model is unsuitable, i.e., not just, e.g., "v is undefined"

5bii **M1:** A correct method for the distance to owner. This statement is sufficient (may be in x). Subtraction from 15 may be seen later

M1: Substitutes their limits into the result from part **a**. Subtraction of lower limit to be seen if limit is not 3

M1: Uses correct logarithmic form of arcosh x

A1: Exact answer only. Accept simplified equivalents, e.g. $5 + \ln 81\sqrt{3}$