C3 PROOF Worksheet A

1 Give a counter-example to prove that each of the following statements is false.

a If $a^2 - b^2 > 0$, where a and b are real, then $a - b > 0$.

b There are no prime numbers divisible by 7.

c If x and y are irrational and $x \neq y$, then xy is irrational.

d For all real values of x, $\cos (90 - |x|)° = \sin x°$.

2 For each statement, either prove that it is true or find a counter-example to prove that it is false.

a There are no prime numbers divisible by 6.

b $(3^n + 2)$ is prime for all positive integer values of n.

c $\sqrt{n}$ is irrational for all positive integers n.

d If a, b and c are integers such that a is divisible by b and b is divisible by c, then a is divisible by c.

3 Use proof by contradiction to prove each of the following statements.

a If n^3 is odd, where n is a positive integer, then n is odd.

b If x is irrational, then $\sqrt{x}$ is irrational.

c If a, b and c are integers and bc is not divisible by a, then b is not divisible by a.

d If $(n^2 - 4n)$ is odd, where n is a positive integer, then n is odd.

e There are no positive integers, m and n, such that $m^2 - n^2 = 6$.

4 Given that x and y are integers and that $(x^2 + y^2)$ is divisible by 4, use proof by contradiction to prove that

a x and y are not both odd,

b x and y are both even.

5 For each statement, either prove that it is true or find a counter-example to prove that it is false.

a If a and b are positive integers and $a \neq b$, then $\log_a b$ is irrational.

b The difference between the squares of any two consecutive odd integers is divisible by 8.

c $(n^2 + 3n + 13)$ is prime for all positive integer values of n.

d For all real values of x and y, $x^2 - 2y(x - y) \geq 0$.

6 **a** Prove that if

$$\sqrt{2} = \frac{p}{q},$$

where p and q are integers, then p must be even.

b Use proof by contradiction to prove that $\sqrt{2}$ is irrational.

C3 PROOF Worksheet B

1 Prove, by counter-example, that each of the following statements is false.

a For all positive real values of x, $\sqrt[3]{x} \leq x$. **(2)**

b For all positive integer values of n, $(n^3 - n + 7)$ is prime. **(2)**

2 Use proof by contradiction to prove that $\sqrt{\pi}$ is irrational.

(You may assume that π is irrational). **(4)**

3 Find a counter-example to prove that the statement

"$15x^2 - 11x + 2 \geq 0$ for all real values of x"

is false. **(4)**

4 **a** Given that $n = 2m + 1$, find and simplify an expression in terms of m for $n^2 + 2n$. **(1)**

b Hence, use proof by contradiction to prove that if $(n^2 + 2n)$ is even, where n is an integer, then n is even. **(5)**

5 **a** Prove that if the equation

$$k \cos x - \operatorname{cosec} x = 0,$$

where k is a constant, has real solutions, then $|k| \geq 2$. **(5)**

b Find the values of x in the interval $0 \leq x \leq 360$ for which

$$3 \cos x° - \operatorname{cosec} x° = 0.$$ **(3)**

6 Use proof by contradiction to prove that there are no positive integers, x and y, such that

$$x^2 - y^2 = 1.$$ **(6)**

7 For each statement, either prove that it is true or find a counter-example to prove that it is false.

a If a and b are irrational and $a \neq b$, then $(a + b)$ is irrational. **(2)**

b If m and n are consecutive odd integers, then $(m + n)$ is divisible by 4. **(3)**

c For all real values of x, $\cos x \leq 1 + \sin x$. **(2)**

8 **a** Show that if $\log_2 3 = \frac{p}{q}$, then

$$2^p = 3^q.$$ **(2)**

b Use proof by contradiction to prove that $\log_2 3$ is irrational. **(4)**

c Prove, by counter-example, that the statement

"if a is rational and b is irrational then $\log_a b$ is irrational"

is false. **(2)**

9 The function f is defined by

$$\mathrm{f} : x \rightarrow \frac{x-2}{4x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

a Find an expression for the inverse function, $\mathrm{f}^{-1}(x)$, and state its domain. **(5)**

b Prove that there are no real values of x for which

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x).$$ **(4)**

C3 PROOF — Answers - Worksheet A

1 **a** e.g. $a = -2, b = 1 \Rightarrow a^2 - b^2 = 4 - 1 = 3 \Rightarrow a^2 - b^2 > 0$
and $a - b = -2 - 1 = -3 \Rightarrow a - b < 0$
[any negative value of a such that $|a| > |b|$]

b 7 7 is prime and divisible by 7 [no other examples]

c e.g. $x = \sqrt{2}, y = 2\sqrt{2} \Rightarrow x$ and y irrational
and $xy = 4$ which is rational [many other examples]

d e.g. $x = -90 \Rightarrow \cos(90 - |x|)° = \cos 0 = 1$
and $\sin x° = \sin(-90°) = -1$ [any –ve x except multiples of 180]

2 **a** true any number divisible by 6 is also divisible by 2 and $\therefore$ not prime

b

n	1	2	3	4	5
$3^n + 2$	5	11	29	83	245

false e.g. $n = 5 \Rightarrow 3^n + 2 = 245$ which is divisible by 5 and $\therefore$ not prime
[many other examples]

c false e.g. $n = 4 \Rightarrow \sqrt{n} = 2$ which is rational [many other examples]

d true b divisible by $c \Rightarrow b = kc,\ k \in \mathbb{Z}$
a divisible by $b \Rightarrow a = lb,\ l \in \mathbb{Z} \Rightarrow a = klc \ \therefore a$ is divisible by c

3 **a** assume n^3 odd and n even, where $n \in \mathbb{Z}^+$
n even $\Rightarrow n = 2m,\ m \in \mathbb{Z}$
$\Rightarrow n^3 = (2m)^3 = 8m^3 = 2(4m^3)$
$4m^3 \in \mathbb{Z} \ \therefore n^3$ even
$\Rightarrow$ contradiction $\therefore n$ odd

b assume x irrational and $\sqrt{x}$ rational
$\sqrt{x}$ rational $\Rightarrow \sqrt{x} = \frac{p}{q},\ p, q \in \mathbb{Z}$
$\Rightarrow x = \frac{p^2}{q^2},\ p^2, q^2 \in \mathbb{Z} \ \therefore x$ rational
$\Rightarrow$ contradiction $\therefore \sqrt{x}$ irrational

c assume bc not divisible by a and b divisible by a where $a, b, c \in \mathbb{Z}$
b divisible by $a \Rightarrow b = ka,\ k \in \mathbb{Z}$
$\Rightarrow bc = kac$ which is divisible by a
$\Rightarrow$ contradiction $\therefore b$ is not divisible by a

d assume $n^2 - 4n$ odd and n even, where $n \in \mathbb{Z}^+$
n even $\Rightarrow n = 2m,\ m \in \mathbb{Z}$
$\Rightarrow n^2 - 4n = (2m)^2 - 4(2m) = 4m^2 - 8m = 2(2m^2 - 4m)$
$2m^2 - 4m \in \mathbb{Z} \ \therefore n^2 - 4n$ even
$\Rightarrow$ contradiction $\therefore n$ odd

e assume $m^2 - n^2 = 6$, where $m, n \in \mathbb{Z}^+$
$m^2 - n^2 = 6 \Rightarrow (m + n)(m - n) = 6$
$m, n \in \mathbb{Z}^+ \Rightarrow (m + n), (m - n) \in \mathbb{Z},\ (m + n) > (m - n)$ and $(m + n) > 0$
$\therefore m + n = 6$ and $m - n = 1$ or $m + n = 3$ and $m - n = 2$
adding $\Rightarrow 2m = 7$ or $2m = 5$
$\Rightarrow m = \frac{7}{2}$ or $m = \frac{5}{2} \Rightarrow m$ not an integer
$\Rightarrow$ contradiction $\therefore$ no positive integer solutions

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4 **a** assume $x^2 + y^2$ divisible by 4 and x, y odd integers

x, y odd $\Rightarrow$ $x = 2m + 1,\ m \in \mathbb{Z}$ and $y = 2n + 1,\ n \in \mathbb{Z}$

$\Rightarrow$ $x^2 + y^2 = (2m + 1)^2 + (2n + 1)^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$

$= 4(m^2 + m + n^2 + n) + 2$

$m^2 + m + n^2 + n \in \mathbb{Z}$ $\therefore$ $x^2 + y^2$ not divisible by 4

$\Rightarrow$ contradiction $\therefore$ x and y not both odd

b assume $x^2 + y^2$ divisible by 4, x odd integer and y even integer

x odd, y even $\Rightarrow$ $x = 2m + 1,\ m \in \mathbb{Z}$ and $y = 2n,\ n \in \mathbb{Z}$

$\Rightarrow$ $x^2 + y^2 = (2m + 1)^2 + (2n)^2$

$= 4m^2 + 4m + 1 + 4n^2$

$= 4(m^2 + m + n^2) + 1$

$m^2 + m + n^2 \in \mathbb{Z}$ $\therefore$ $x^2 + y^2$ not divisible by 4

$\Rightarrow$ contradiction $\therefore$ x odd and y even not possible

same argument applies with x even and y odd

part **a** shows x and y can't both be odd

$\therefore$ x and y both even

5 **a** false e.g. $a = 2, b = 4$ $\Rightarrow$ $\log_a b = 2$ which is rational

[many other examples]

b true $(2n + 1)$ and $(2n + 3)$, $n \in \mathbb{Z}$ represent any two consecutive odd integers

$(2n + 3)^2 - (2n + 1)^2 = 4n^2 + 12n + 9 - (4n^2 + 4n + 1)$

$= 8n + 8$

$= 8(n + 1)$

$n + 1 \in \mathbb{Z}$ $\therefore$ difference is divisible by 8

c false e.g. $n = 13$ $\Rightarrow$ $n^2 + 3n + 13 = 13(13 + 3 + 1)$ which is divisible by 13

[many other examples]

d true $x^2 - 2y(x - y) = x^2 - 2xy + 2y^2$

$= x^2 - 2xy + y^2 + y^2$

$= (x - y)^2 + y^2$

for real x and y, $(x - y)^2 \geq 0$ and $y^2 \geq 0$ $\therefore$ $x^2 - 2y(x - y) \geq 0$

6 **a** $\sqrt{2} = \frac{p}{q},\ p, q \in \mathbb{Z}$ $\Rightarrow$ $2 = \frac{p^2}{q^2}$ $\Rightarrow$ $p^2 = 2q^2$

$\Rightarrow$ p^2 even $\Rightarrow$ p even

b assume $\sqrt{2}$ rational $\Rightarrow$ $\sqrt{2} = \frac{p}{q},\ p, q \in \mathbb{Z}$ and p, q co-prime

part **a** $\Rightarrow$ p even $\Rightarrow$ $p = 2n,\ n \in \mathbb{Z}$

$\Rightarrow$ $(2n)^2 = 2q^2$

$\Rightarrow$ $q^2 = 2n^2$

$\Rightarrow$ q^2 even $\Rightarrow$ q even

$\Rightarrow$ p and q both even $\therefore$ not co-prime

$\Rightarrow$ contradiction $\therefore$ $\sqrt{2}$ is irrational

C3 PROOF — *Answers - Worksheet B*

1 **a** e.g. $x = \frac{1}{8} \Rightarrow \sqrt[3]{x} = \frac{1}{2}, \ \frac{1}{2} > \frac{1}{8}$

[any value of x in the interval $0 < x < 1$]

b e.g. $n = 7 \Rightarrow n^3 - n + 7 = 7(49 - 1 + 1)$ which is divisible by 7

[many other examples]

2 assume $\sqrt{\pi}$ is rational $\Rightarrow \sqrt{\pi} = \frac{p}{q}, \ p, q \in \mathbb{Z}$

$\Rightarrow \pi = \frac{p^2}{q^2}, \ p^2, q^2 \in \mathbb{Z} \ \therefore \pi$ rational

$\Rightarrow$ contradiction $\therefore \sqrt{\pi}$ irrational

3 consider $15x^2 - 11x + 2 < 0$

$\Rightarrow (5x - 2)(3x - 1) < 0$

$\Rightarrow \frac{1}{3} < x < \frac{2}{5}$

e.g. $x = 0.35 \Rightarrow 15x^2 - 11x + 2 = -0.0125, \ -0.0125 < 0$

[any value of x in the interval $\frac{1}{3} < x < \frac{2}{5}$]

4 **a** $n^2 + 2n = (2m + 1)^2 + 2(2m + 1)$

$= 4m^2 + 4m + 1 + 4m + 2$

$= 4m^2 + 8m + 3$

b assume $n^2 + 2n$ even and n odd, where $n \in \mathbb{Z}$

n odd $\Rightarrow n = 2m + 1, \ m \in \mathbb{Z}$

$\Rightarrow n^2 + 2n = 4m^2 + 8m + 3 = 2(2m^2 + 4m + 1) + 1$

$2m^2 + 4m + 1 \in \mathbb{Z} \ \therefore n^2 + 2n$ odd

$\Rightarrow$ contradiction $\therefore n$ even

5 **a** $k \cos x - \operatorname{cosec} x = 0 \Rightarrow k \cos x = \frac{1}{\sin x}$

$\Rightarrow k \sin x \cos x = 1$

$\Rightarrow \frac{1}{2} k \sin 2x = 1$

$\Rightarrow \sin 2x = \frac{2}{k}$

$|\sin 2x| \le 1 \Rightarrow \left|\frac{2}{k}\right| \le 1$

$\Rightarrow |k| \ge 2$

b $3 \cos x - \operatorname{cosec} x = 0 \Rightarrow \sin 2x = \frac{2}{3}$

$2x = 41.810, 180 - 41.810, 360 + 41.810, 540 - 41.810$

$2x = 41.810, 138.190, 401.810, 498.190$

$x = 20.9, 69.1, 200.9, 249.1$ (1dp)

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6 assume $x^2 - y^2 = 1$, where $x, y \in \mathbb{Z}^+$

$x^2 - y^2 = 1 \Rightarrow (x + y)(x - y) = 1$

$x, y \in \mathbb{Z}^+ \Rightarrow (x + y), (x - y) \in \mathbb{Z}$ and $(x + y) > 0$

$\therefore$ $x + y = 1$ and $x - y = 1$

adding $\Rightarrow 2x = 2$

$\Rightarrow x = 1$

$\Rightarrow y = 0$

$\Rightarrow$ contradiction $\therefore$ no positive integer solutions

7 **a** false e.g. $a = \sqrt{2}$, $b = 1 - \sqrt{2}$ $\Rightarrow$ a and b irrational

and $a + b = 1$ which is rational

[many other examples]

b true m, n consecutive odd integers $\Rightarrow$ $m = 2a + 1, n = 2a + 3,\ a \in \mathbb{Z}$

$\Rightarrow m + n = 2a + 1 + 2a + 3 = 4a + 4 = 4(a + 1)$

$a + 1 \in \mathbb{Z}$ $\therefore$ $m + n$ divisible by 4

c false e.g. $x = \frac{5\pi}{3} \Rightarrow \cos x = \frac{1}{2}$ and $1 + \sin x = 1 - \frac{\sqrt{3}}{2}$, $\frac{1}{2} > 1 - \frac{\sqrt{3}}{2}$

[any value of x of the form $2n\pi + y,\ n \in \mathbb{Z},\ -\frac{\pi}{2} < y < 0$]

8 **a** $\log_2 3 = \frac{p}{q} \Rightarrow 2^{\frac{p}{q}} = 3$

$\Rightarrow (2^{\frac{p}{q}})^q = 3^q$

$\Rightarrow 2^p = 3^q$

b assume $\log_2 3$ is rational $\Rightarrow \log_2 3 = \frac{p}{q},\ p, q \in \mathbb{Z},\ q \neq 0$

$\Rightarrow 2^p = 3^q$

2 and 3 are co-prime $\Rightarrow p = q = 0$

$\Rightarrow$ contradiction $\therefore$ $\log_2 3$ is irrational

c e.g. $a = 2, b = \sqrt{2}$ $\Rightarrow$ a rational and b irrational

and $\log_a b = \frac{1}{2}$ which is rational

[many other examples]

9 **a** $y = \frac{x-2}{4x}$ swap $x = \frac{y-2}{4y}$

$4xy = y - 2$

$y(4x - 1) = -2$

$y = \frac{2}{1-4x}$

$f^{-1}(x) = \frac{2}{1-4x},\ x \in \mathbb{R},\ x \neq \frac{1}{4}$

b $f(x) = f^{-1}(x) \Rightarrow \frac{x-2}{4x} = \frac{2}{1-4x}$

$\Rightarrow (x - 2)(1 - 4x) = 8x$

$\Rightarrow 4x^2 - x + 2 = 0$

$b^2 - 4ac = 1 - 32 = -31$

$b^2 - 4ac < 0 \Rightarrow$ no real roots

$\therefore$ no real values of x for which $f(x) = f^{-1}(x)$