C2 DIFFERENTIATION — Worksheet A

1 In each case, find any values of x for which $\frac{dy}{dx} = 0$.

a $y = x^2 + 6x$ **b** $y = 4x^2 + 2x + 1$ **c** $y = x^3 - 12x$ **d** $y = 4 + 9x^2 - x^3$

e $y = x^3 - 5x^2 + 3x$ **f** $y = x + \frac{9}{x}$ **g** $y = (x^2 + 3)(x - 3)$ **h** $y = x^{\frac{1}{2}} - 2x$

2 Find the set of values of x for which f(x) is increasing when

a $f(x) \equiv 2x^2 + 2x + 1$ **b** $f(x) \equiv 3x^2 - 2x^3$ **c** $f(x) \equiv 3x^3 - x - 7$

d $f(x) \equiv x^3 + 6x^2 - 15x + 8$ **e** $f(x) \equiv x(x - 6)^2$ **f** $f(x) \equiv 2x + \frac{8}{x}$

3 Find the set of values of x for which f(x) is decreasing when

a $f(x) \equiv x^3 + 2x^2 + 1$ **b** $f(x) \equiv 5 + 27x - x^3$ **c** $f(x) \equiv (x^2 - 2)(2x - 1)$

4 $$f(x) \equiv x^3 + kx^2 + 3.$$

Given that $(x + 1)$ is a factor of f(x),

a find the value of the constant k,

b find the set of values of x for which f(x) is increasing.

5 Find the coordinates of any stationary points on each curve.

a $y = x^2 + 2x$ **b** $y = 5x^2 - 4x + 1$ **c** $y = x^3 - 3x + 4$

d $y = 4x^3 + 3x^2 + 2$ **e** $y = 2x + 3 + \frac{8}{x}$ **f** $y = x^3 - 9x^2 - 21x + 11$

g $y = \frac{1}{x} - 4x^2$ **h** $y = 2x^{\frac{3}{2}} - 6x$ **i** $y = 9x^{\frac{2}{3}} - 2x + 5$

6 Find the coordinates of any stationary points on each curve. By evaluating $\frac{d^2y}{dx^2}$ at each stationary point, determine whether it is a maximum or minimum point.

a $y = 5 + 4x - x^2$ **b** $y = x^3 - 3x$ **c** $y = x^3 + 9x^2 - 8$

d $y = x^3 - 6x^2 - 36x + 15$ **e** $y = x^4 - 8x^2 - 2$ **f** $y = 9x + \frac{4}{x}$

g $y = x - 6x^{\frac{1}{2}}$ **h** $y = 3 - 8x + 7x^2 - 2x^3$ **i** $y = \frac{x^4 + 16}{2x^2}$

7 Find the coordinates of any stationary points on each curve and determine whether each stationary point is a maximum, minimum or point of inflexion.

a $y = x^2 - x^3$ **b** $y = x^3 + 3x^2 + 3x$ **c** $y = x^4 - 2$

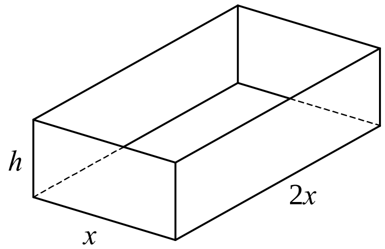
d $y = 4 - 12x + 6x^2 - x^3$ **e** $y = x^2 + \frac{16}{x}$ **f** $y = x^4 + 4x^3 - 1$

8 Sketch each of the following curves showing the coordinates of any turning points.

a $y = x^3 + 3x^2$ **b** $y = x + \frac{1}{x}$ **c** $y = x^3 - 3x^2 + 3x - 1$

d $y = 3x - 4x^{\frac{1}{2}}$ **e** $y = x^3 + 4x^2 - 3x - 5$ **f** $y = (x^2 - 2)(x^2 - 6)$

1



The diagram shows a baking tin in the shape of an open-topped cuboid made from thin metal sheet. The base of the tin measures x cm by $2x$ cm, the height of the tin is h cm and the volume of the tin is 4000 cm^3.

a Find an expression for h in terms of x.

b Show that the area of metal sheet used to make the tin, A cm^2, is given by

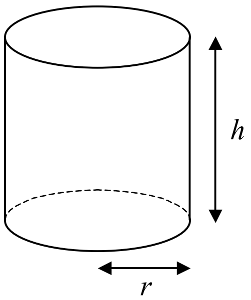
$$A = 2x^2 + \frac{12\,000}{x}.$$

c Use differentiation to find the value of x for which A is a minimum.

d Find the minimum value of A.

e Show that your value of A is a minimum.

2



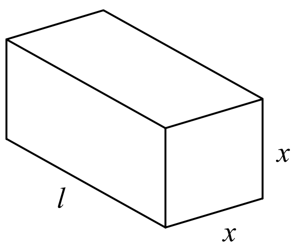
The diagram shows a closed plastic cylinder used for making compost. The radius of the base and the height of the cylinder are r cm and h cm respectively and the surface area of the cylinder is 30 000 cm^2.

a Show that the volume of the cylinder, V cm^3, is given by

$$V = 15\,000r - \pi r^3.$$

b Find the maximum volume of the cylinder and show that your value is a maximum.

3



The diagram shows a square prism of length l cm and cross-section x cm by x cm.
Given that the surface area of the prism is k cm^2, where k is a constant,

a show that $l = \frac{k - 2x^2}{4x}$,

b use calculus to prove that the maximum volume of the prism occurs when it is a cube.

1 $$f(x) \equiv 2x^3 + 5x^2 - 1.$$

a Find $f'(x)$.

b Find the set of values of x for which $f(x)$ is increasing.

2 The curve C has the equation $y = x^3 - x^2 + 2x - 4$.

a Find an equation of the tangent to C at the point $(1, -2)$. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

b Prove that the curve C has no stationary points.

3 A curve has the equation $y = \sqrt{x} + \frac{4}{x}$.

a Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

b Find the coordinates of the stationary point of the curve and determine its nature.

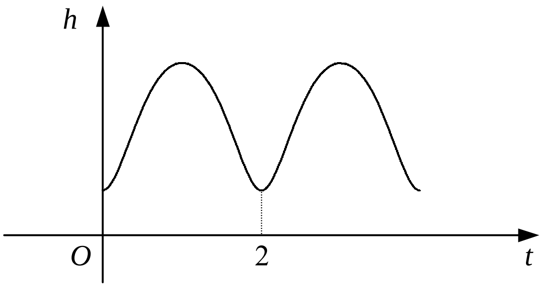
4 $$f(x) \equiv x^3 + 6x^2 + 9x.$$

a Find the coordinates of the points where the curve $y = f(x)$ meets the x-axis.

b Find the set of values of x for which $f(x)$ is decreasing.

c Sketch the curve $y = f(x)$, showing the coordinates of any stationary points.

5



The graph shows the height, h cm, of the letters on a website advert t seconds after the advert appears on the screen.

For t in the interval $0 \le t \le 2$, h is given by the equation

$$h = 2t^4 - 8t^3 + 8t^2 + 1.$$

For larger values of t, the variation of h over this interval is repeated every 2 seconds.

a Find $\frac{dh}{dt}$ for t in the interval $0 \le t \le 2$.

b Find the rate at which the height of the letters is increasing when $t = 0.25$

c Find the maximum height of the letters.

6 The curve C has the equation $y = x^3 + 3kx^2 - 9k^2x$, where k is a non-zero constant.

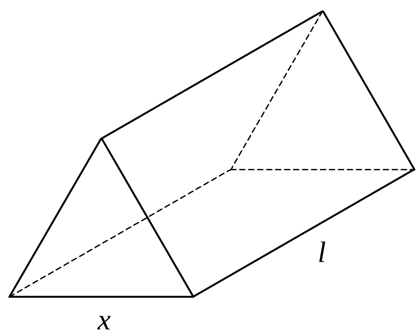
a Show that C is stationary when

$$x^2 + 2kx - 3k^2 = 0.$$

b Hence, show that C is stationary at the point with coordinates $(k, -5k^3)$.

c Find, in terms of k, the coordinates of the other stationary point on C.

7



The diagram shows a solid triangular prism. The cross-section of the prism is an equilateral triangle of side x cm and the length of the prism is l cm.

Given that the volume of the prism is 250 cm^3,

a find an expression for l in terms of x,

b show that the surface area of the prism, A cm^2, is given by

$$A = \frac{\sqrt{3}}{2}(x^2 + \frac{2000}{x}).$$

Given that x can vary,

c find the value of x for which A is a minimum,

d find the minimum value of A in the form $k\sqrt{3}$,

e justify that the value you have found is a minimum.

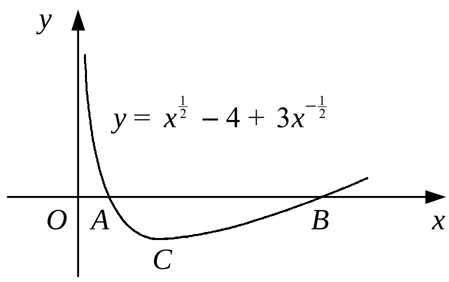
8 $$f(x) \equiv x^3 + 4x^2 + kx + 1.$$

a Find the set of values of the constant k for which the curve $y = f(x)$ has two stationary points.

Given that $k = -3$,

b find the coordinates of the stationary points of the curve $y = f(x)$.

9



The diagram shows the curve with equation $y = x^{\frac{1}{2}} - 4 + 3x^{-\frac{1}{2}}$. The curve crosses the x-axis at the points A and B and has a minimum point at C.

a Find the coordinates of A and B.

b Find the coordinates of C, giving its y-coordinate in the form $a\sqrt{3} + b$, where a and b are integers.

10 $$f(x) = x^3 - 3x^2 + 4.$$

a Show that $(x + 1)$ is a factor of f(x).

b Fully factorise f(x).

c Hence state, with a reason, the coordinates of one of the turning points of the curve $y = f(x)$.

d Using differentiation, find the coordinates of the other turning point of the curve $y = f(x)$.

1 $$f(x) \equiv 7 + 24x + 3x^2 - x^3.$$

a Find $f'(x)$. **(2)**

b Find the set of values of x for which $f(x)$ is increasing. **(4)**

2 The curve with equation $y = x^3 + ax^2 - 24x + b$, where a and b are constants, passes through the point $P(-2, 30)$.

a Show that $4a + b + 10 = 0$. **(2)**

Given also that P is a stationary point of the curve,

b find the values of a and b, **(4)**

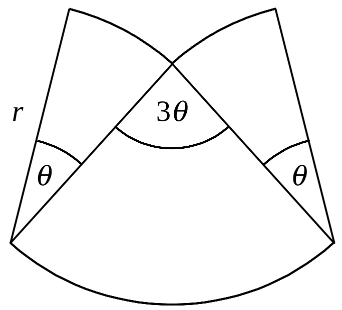
c find the coordinates of the other stationary point on the curve. **(3)**

3 $$f(x) \equiv x^2 + \frac{16}{x}, \quad x \neq 0.$$

a Find $f'(x)$. **(2)**

b Find the coordinates of the stationary point of the curve $y = f(x)$ and determine its nature. **(6)**

4



The diagram shows a design to be used on a new brand of cat-food. The design consists of three circular sectors, each of radius r cm. The angle of two of the sectors is θ radians and the angle of the third sector is 3θ radians as shown.

Given that the area of the design is 25 cm^2,

a show that $\theta = \frac{10}{r^2}$, **(3)**

b find the perimeter of the design, P cm, in terms of r. **(3)**

Given that r can vary,

c find the value of r for which P takes it minimum value, **(4)**

d find the minimum value of P, **(1)**

e justify that the value you have found is a minimum. **(2)**

5 The curve C has the equation

$$y = 2x - x^{\frac{3}{2}}, \quad x \geq 0.$$

a Find the coordinates of any points where C meets the x-axis. **(3)**

b Find the x-coordinate of the stationary point on C and determine whether it is a maximum or a minimum point. **(6)**

c Sketch the curve C. **(2)**

6 The curve $y = x^3 - 3x + 1$ is stationary at the points P and Q.

a Find the coordinates of the points P and Q. **(5)**

b Find the length of PQ in the form $k\sqrt{5}$. **(3)**

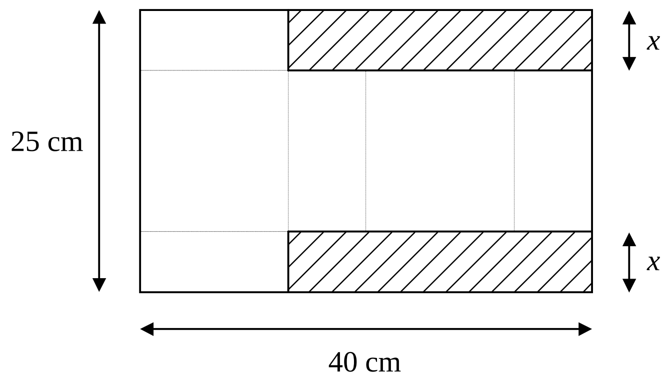
7 $$f(x) \equiv 2x - 5 + \frac{2}{x}, \quad x \neq 0.$$

a Solve the equation $f(x) = 0$. **(4)**

b Solve the equation $f'(x) = 0$. **(4)**

c Sketch the curve $y = f(x)$, showing the coordinates of any turning points and of any points where the curve crosses the coordinate axes. **(3)**

8



Two identical rectangles of width x cm are removed from a rectangular piece of card measuring 25 cm by 40 cm as shown in the diagram above. The remaining card is the net of a cuboid of height x cm.

a Find expressions in terms of x for the length and width of the base of the cuboid formed from the net. **(3)**

b Show that the volume of the cuboid is $(2x^3 - 65x^2 + 500x)$ cm^3. **(2)**

c Find the value of x for which the volume of the cuboid is a maximum. **(5)**

d Find the maximum volume of the cuboid and show that it is a maximum. **(3)**

9 **a** Find the coordinates of the stationary points on the curve

$$y = 2 + 9x + 3x^2 - x^3.$$ **(6)**

b Determine whether each stationary point is a maximum or minimum point. **(2)**

c State the set of values of k for which the equation

$$2 + 9x + 3x^2 - x^3 = k$$

has three solutions. **(2)**

10 $$f(x) = 4x^3 + ax^2 - 12x + b.$$

Given that a and b are constants and that when $f(x)$ is divided by $(x + 1)$ there is a remainder of 15,

a find the value of $(a + b)$. **(2)**

Given also that when $f(x)$ is divided by $(x - 2)$ there is a remainder of 42,

b find the values of a and b, **(3)**

c find the coordinates of the stationary points of the curve $y = f(x)$. **(6)**

1 **a** $\frac{dy}{dx} = 2x + 6$
$2x + 6 = 0$
$x = -3$

b $\frac{dy}{dx} = 8x + 2$
$8x + 2 = 0$
$x = -\frac{1}{4}$

c $\frac{dy}{dx} = 3x^2 - 12$
$3x^2 - 12 = 0$
$x^2 = 4$
$x = \pm 2$

d $\frac{dy}{dx} = 18x - 3x^2$
$18x - 3x^2 = 0$
$3x(6 - x) = 0$
$x = 0, 6$

e $\frac{dy}{dx} = 3x^2 - 10x + 3$
$3x^2 - 10x + 3 = 0$
$(3x - 1)(x - 3) = 0$
$x = \frac{1}{3}, 3$

f $\frac{dy}{dx} = 1 - 9x^{-2}$
$1 - 9x^{-2} = 0$
$x^2 = 9$
$x = \pm 3$

g $y = x^3 - 3x^2 + 3x - 9$
$\frac{dy}{dx} = 3x^2 - 6x + 3$
$3x^2 - 6x + 3 = 0$
$3(x - 1)^2 = 0$
$x = 1$

h $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - 2$
$\frac{1}{2}x^{-\frac{1}{2}} - 2 = 0$
$x^{-\frac{1}{2}} = 4$
$x = \frac{1}{16}$

2 **a** $f'(x) = 4x + 2$
$\therefore\ 4x + 2 \geq 0$
$x \geq -\frac{1}{2}$

b $f'(x) = 6x - 6x^2$
$\therefore\ 6x - 6x^2 \geq 0$
$6x(1 - x) \geq 0$
$0 \leq x \leq 1$

c $f'(x) = 9x^2 - 1$
$\therefore\ 9x^2 - 1 \geq 0$
$x^2 \geq \frac{1}{9}$
$x \leq -\frac{1}{3}$ and $x \geq \frac{1}{3}$

d $f'(x) = 3x^2 + 12x - 15$
$\therefore\ 3x^2 + 12x - 15 \geq 0$
$3(x + 5)(x - 1) \geq 0$
$x \leq -5$ and $x \geq 1$

e $f(x) = x^3 - 12x^2 + 36x$
$f'(x) = 3x^2 - 24x + 36$
$\therefore\ 3x^2 - 24x + 36 \geq 0$
$3(x - 2)(x - 6) \geq 0$
$x \leq 2$ and $x \geq 6$

f $f'(x) = 2 - 8x^{-2}$
$\therefore\ 2 - 8x^{-2} \geq 0$
$x^2 \geq 4$
$x \leq -2$ and $x \geq 2$

3 **a** $f'(x) = 3x^2 + 4x$
$\therefore\ 3x^2 + 4x \leq 0$
$x(3x + 4) \leq 0$
$-\frac{4}{3} \leq x \leq 0$

b $f'(x) = 27 - 3x^2$
$\therefore\ 27 - 3x^2 \leq 0$
$x^2 \geq 9$
$x \leq -3$ and $x \geq 3$

c $f(x) = 2x^3 - x^2 - 4x + 2$
$f'(x) = 6x^2 - 2x - 4$
$\therefore\ 6x^2 - 2x - 4 \leq 0$
$2(3x + 2)(x - 1) \leq 0$
$-\frac{2}{3} \leq x \leq 1$

4 **a** $(x + 1)$ factor $\therefore\ f(-1) = 0$
$\therefore\ -1 + k + 3 = 0$
$k = -2$

b $f'(x) = 3x^2 - 4x$
$\therefore\ 3x^2 - 4x \geq 0$
$x(3x - 4) \geq 0$
$x \leq 0$ and $x \geq \frac{4}{3}$

5 **a** $\frac{dy}{dx} = 2x + 2$
SP: $2x + 2 = 0$
$x = -1$
$\therefore$ $(-1, -1)$

b $\frac{dy}{dx} = 10x - 4$
SP: $10x - 4 = 0$
$x = \frac{2}{5}$
$\therefore$ $(\frac{2}{5}, \frac{1}{5})$

c $\frac{dy}{dx} = 3x^2 - 3$
SP: $3x^2 - 3 = 0$
$x^2 = 1$
$x = \pm 1$
$\therefore$ $(-1, 6), (1, 2)$

d $\frac{dy}{dx} = 12x^2 + 6x$
SP: $12x^2 + 6x = 0$
$6x(2x + 1) = 0$
$x = -\frac{1}{2}, 0$
$\therefore$ $(-\frac{1}{2}, \frac{9}{4}), (0, 2)$

e $\frac{dy}{dx} = 2 - 8x^{-2}$
SP: $2 - 8x^{-2} = 0$
$x^2 = 4$
$x = \pm 2$
$\therefore$ $(-2, -5), (2, 11)$

f $\frac{dy}{dx} = 3x^2 - 18x - 21$
SP: $3x^2 - 18x - 21 = 0$
$3(x + 1)(x - 7) = 0$
$x = -1, 7$
$\therefore$ $(-1, 22), (7, -234)$

g $\frac{dy}{dx} = -x^{-2} - 8x$
SP: $-x^{-2} - 8x = 0$
$x^3 = -\frac{1}{8}$
$x = -\frac{1}{2}$
$\therefore$ $(-\frac{1}{2}, -3)$

h $\frac{dy}{dx} = 3x^{\frac{1}{2}} - 6$
SP: $3x^{\frac{1}{2}} - 6 = 0$
$x^{\frac{1}{2}} = 2$
$x = 4$
$\therefore$ $(4, -8)$

i $\frac{dy}{dx} = 6x^{-\frac{1}{3}} - 2$
SP: $6x^{-\frac{1}{3}} - 2 = 0$
$x^{-\frac{1}{3}} = 3$
$x = \frac{1}{27}$
$\therefore$ $(\frac{1}{27}, 5\frac{25}{27})$

6 **a** $\frac{dy}{dx} = 4 - 2x$
SP: $4 - 2x = 0$
$x = 2$
$\frac{d^2y}{dx^2} = -2$
$(2, 9)$: max

b $\frac{dy}{dx} = 3x^2 - 3$
SP: $3x^2 - 3 = 0$
$x^2 = 1$
$x = \pm 1$
$\frac{d^2y}{dx^2} = 6x$
$(-1, 2)$: $\frac{d^2y}{dx^2} = -6$, max
$(1, -2)$: $\frac{d^2y}{dx^2} = 6$, min

c $\frac{dy}{dx} = 3x^2 + 18x$
SP: $3x^2 + 18x = 0$
$3x(x + 6) = 0$
$x = -6, 0$
$\frac{d^2y}{dx^2} = 6x + 18$
$(-6, 100)$: $\frac{d^2y}{dx^2} = -18$, max
$(0, -8)$: $\frac{d^2y}{dx^2} = 18$, min

d $\frac{dy}{dx} = 3x^2 - 12x - 36$
SP: $3x^2 - 12x - 36 = 0$
$3(x + 2)(x - 6) = 0$
$x = -2, 6$
$\frac{d^2y}{dx^2} = 6x - 12$
$(-2, 55)$: $\frac{d^2y}{dx^2} = -24$, max
$(6, -201)$: $\frac{d^2y}{dx^2} = 24$, min

e $\frac{dy}{dx} = 4x^3 - 16x$
SP: $4x^3 - 16x = 0$
$4x(x^2 - 4) = 0$
$x = 0, \pm 2$
$\frac{d^2y}{dx^2} = 12x^2 - 16$
$(-2, -18)$: $\frac{d^2y}{dx^2} = 32$, min
$(0, -2)$: $\frac{d^2y}{dx^2} = -16$, max
$(2, -18)$: $\frac{d^2y}{dx^2} = 32$, min

f $\frac{dy}{dx} = 9 - 4x^{-2}$
SP: $9 - 4x^{-2} = 0$
$x^2 = \frac{4}{9}$
$x = \pm \frac{2}{3}$
$\frac{d^2y}{dx^2} = 8x^{-3}$
$(-\frac{2}{3}, -12)$: $\frac{d^2y}{dx^2} = -27$, max
$(\frac{2}{3}, 12)$: $\frac{d^2y}{dx^2} = 27$, min

g $\frac{dy}{dx} = 1 - 3x^{-\frac{1}{2}}$

SP: $1 - 3x^{-\frac{1}{2}} = 0$

$x^{-\frac{1}{2}} = \frac{1}{3}$

$x = 9$

$\frac{d^2y}{dx^2} = \frac{3}{2}x^{-\frac{3}{2}}$

$(9, -9)$: $\frac{d^2y}{dx^2} = \frac{1}{18}$, min

h $\frac{dy}{dx} = -8 + 14x - 6x^2$

SP: $-8 + 14x - 6x^2 = 0$

$-2(3x - 4)(x - 1) = 0$

$x = 1, \frac{4}{3}$

$\frac{d^2y}{dx^2} = 14 - 12x$

$(1, 0)$: $\frac{d^2y}{dx^2} = 2$, min

$(\frac{4}{3}, \frac{1}{27})$: $\frac{d^2y}{dx^2} = -2$, max

i $y = \frac{1}{2}x^2 + 8x^{-2}$

$\frac{dy}{dx} = x - 16x^{-3}$

SP: $x - 16x^{-3} = 0$

$x^4 = 16$

$x = \pm 2$

$\frac{d^2y}{dx^2} = 1 + 48x^{-4}$

$(-2, 4)$: $\frac{d^2y}{dx^2} = 4$, min

$(2, 4)$: $\frac{d^2y}{dx^2} = 4$, min

7 **a** $\frac{dy}{dx} = 2x - 3x^2$

SP: $2x - 3x^2 = 0$

$x(2 - 3x) = 0$

$x = 0, \frac{2}{3}$

$\frac{d^2y}{dx^2} = 2 - 6x$

$(0, 0)$: $\frac{d^2y}{dx^2} = 2$, min

$(\frac{2}{3}, \frac{4}{27})$: $\frac{d^2y}{dx^2} = -2$, max

b $\frac{dy}{dx} = 3x^2 + 6x + 3$

SP: $3x^2 + 6x + 3 = 0$

$3(x + 1)^2 = 0$

$x = -1$

$\frac{d^2y}{dx^2} = 6x + 6$

$(-1, -1)$: $\frac{d^2y}{dx^2} = 0$

x	< -1	-1	> -1
$\frac{dy}{dx}$	+	0	+

$\therefore$ $(-1, -1)$: point of inflexion

c $\frac{dy}{dx} = 4x^3$

SP: $4x^3 = 0$

$x = 0$

$\frac{d^2y}{dx^2} = 12x^2$

$(0, -2)$: $\frac{d^2y}{dx^2} = 0$

x	< 0	0	> 0
$\frac{dy}{dx}$	–	0	+

$\therefore$ $(0, -2)$: min

d $\frac{dy}{dx} = -12 + 12x - 3x^2$

SP: $-12 + 12x - 3x^2 = 0$

$-3(x - 2)^2 = 0$

$x = 2$

$\frac{d^2y}{dx^2} = 12 - 6x$

$(2, -4)$: $\frac{d^2y}{dx^2} = 0$

x	< 2	2	> 2
$\frac{dy}{dx}$	–	0	–

$\therefore$ $(2, -4)$: point of inflexion

e $\frac{dy}{dx} = 2x - 16x^{-2}$

SP: $2x - 16x^{-2} = 0$

$x^3 = 8$

$x = 2$

$\frac{d^2y}{dx^2} = 2 + 32x^{-3}$

$(2, 12)$: $\frac{d^2y}{dx^2} = 6$, min

f $\frac{dy}{dx} = 4x^3 + 12x^2$

SP: $4x^3 + 12x^2 = 0$

$4x^2(x + 3) = 0$

$x = -3, 0$

$\frac{d^2y}{dx^2} = 12x^2 + 24x$

$(-3, -28)$: $\frac{d^2y}{dx^2} = 36$, min

$(0, -1)$: $\frac{d^2y}{dx^2} = 0$

x	$-3 < x < 0$	0	> 0
$\frac{dy}{dx}$	+	0	+

$\therefore$ $(0, -1)$: point of inflexion

8 **a** $\frac{dy}{dx} = 3x^2 + 6x$

SP: $3x^2 + 6x = 0$

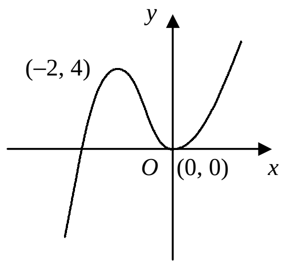
$3x(x + 2) = 0$

$x = -2, 0$

$\frac{d^2y}{dx^2} = 6x + 6$

$(-2, 4)$: $\frac{d^2y}{dx^2} = -6$, max

$(0, 0)$: $\frac{d^2y}{dx^2} = 6$, min



b $\frac{dy}{dx} = 1 - x^{-2}$

SP: $1 - x^{-2} = 0$

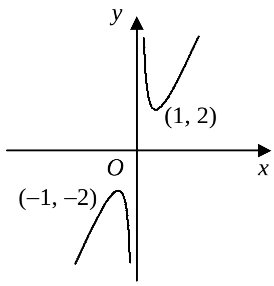
$x^2 = 1$

$x = \pm 1$

$\frac{d^2y}{dx^2} = 2x^{-3}$

$(-1, -2)$: $\frac{d^2y}{dx^2} = -2$, max

$(1, 2)$: $\frac{d^2y}{dx^2} = 2$, min



c $\frac{dy}{dx} = 3x^2 - 6x + 3$

SP: $3x^2 - 6x + 3 = 0$

$3(x - 1)^2 = 0$

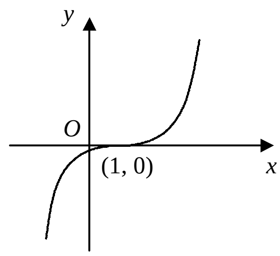
$x = 1$

$\frac{d^2y}{dx^2} = 6x - 6$

$(1, 0)$: $\frac{d^2y}{dx^2} = 0$

x	< 1	1	> 1
$\frac{dy}{dx}$	+	0	+

$\therefore$ $(1, 0)$: point of inflexion



d $\frac{dy}{dx} = 3 - 2x^{-\frac{1}{2}}$

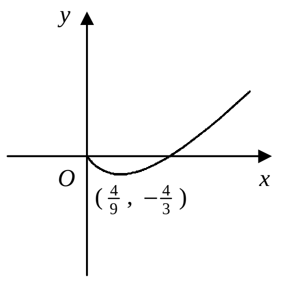
SP: $3 - 2x^{-\frac{1}{2}} = 0$

$x^{-\frac{1}{2}} = \frac{3}{2}$

$x = \frac{4}{9}$

$\frac{d^2y}{dx^2} = x^{-\frac{3}{2}}$

$(\frac{4}{9}, -\frac{4}{3})$: $\frac{d^2y}{dx^2} = \frac{27}{8}$, min



e $\frac{dy}{dx} = 3x^2 + 8x - 3$

SP: $3x^2 + 8x - 3 = 0$

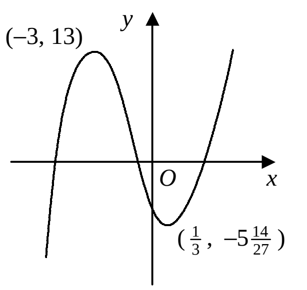
$(3x - 1)(x + 3) = 0$

$x = -3, \frac{1}{3}$

$\frac{d^2y}{dx^2} = 6x + 8$

$(-3, 13)$: $\frac{d^2y}{dx^2} = -10$, max

$(\frac{1}{3}, -5\frac{14}{27})$: $\frac{d^2y}{dx^2} = 10$, min



f $y = x^4 - 8x^2 + 12$

$\frac{dy}{dx} = 4x^3 - 16x$

SP: $4x^3 - 16x = 0$

$4x(x + 2)(x - 2) = 0$

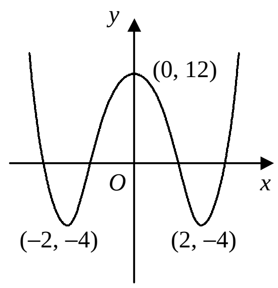
$x = -2, 0, 2$

$\frac{d^2y}{dx^2} = 12x^2 - 16$

$(-2, -4)$: $\frac{d^2y}{dx^2} = 32$, min

$(0, 12)$: $\frac{d^2y}{dx^2} = -16$, max

$(2, -4)$: $\frac{d^2y}{dx^2} = 32$, min



1 **a** volume $= 2x^2h = 4000$

$\therefore\ h = \frac{2000}{x^2}$

b $A = 2x^2 + 2(2xh) + 2(xh)$

$= 2x^2 + 6xh$

$= 2x^2 + (6x \times \frac{2000}{x^2})$

$= 2x^2 + \frac{12\,000}{x}$

c $\frac{dA}{dx} = 4x - 12\,000x^{-2}$

SP: $4x - 12\,000x^{-2} = 0$

$x^3 = 3000$

$x = \sqrt[3]{3000} = 14.4$ (3sf)

d min $A = 1250$ (3sf)

e $\frac{d^2A}{dx^2} = 4 + 24\,000x^{-3}$

when $x = \sqrt[3]{3000}$, $\frac{d^2A}{dx^2} = 12$

$\frac{d^2A}{dx^2} > 0\ \therefore$ minimum

2 **a** S.A. $= 2\pi r^2 + 2\pi rh = 30\,000$

$\therefore\ \pi rh = 15\,000 - \pi r^2$

$h = \frac{15\,000}{\pi r} - r$

$V = \pi r^2 h$

$= \pi r^2(\frac{15\,000}{\pi r} - r)$

$= 15\,000r - \pi r^3$

b $\frac{dV}{dr} = 15\,000 - 3\pi r^2$

SP: $15\,000 - 3\pi r^2 = 0$

$r^2 = \frac{5000}{\pi}$

$r = \sqrt{\frac{5000}{\pi}}$ [= 39.9 (3sf)]

max volume = 399 000 cm^3 (3sf)

$\frac{d^2V}{dr^2} = -6\pi r$

when $r = \sqrt{\frac{5000}{\pi}}$, $\frac{d^2V}{dr^2} = -752$

$\frac{d^2V}{dr^2} < 0\ \therefore$ maximum

3 **a** S.A. $= 2x^2 + 4xl = k$

$\therefore\ 4xl = k - 2x^2$

$l = \frac{k - 2x^2}{4x}$

b $V = x^2 l$

$= x^2 \times \frac{k - 2x^2}{4x}$

$= \frac{1}{4}kx - \frac{1}{2}x^3$

$\frac{dV}{dx} = \frac{1}{4}k - \frac{3}{2}x^2$

SP: $\frac{1}{4}k - \frac{3}{2}x^2 = 0$

$x^2 = \frac{1}{6}k$

$x = \sqrt{\frac{k}{6}}$

$\frac{d^2V}{dx^2} = -3x$

when $x = \sqrt{\frac{k}{6}}$, $\frac{d^2V}{dx^2} < 0\ \therefore$ maximum

$l = \frac{k - \frac{1}{3}k}{4\sqrt{\frac{k}{6}}} = \frac{2}{3}k \times \frac{1}{4} \times \sqrt{\frac{6}{k}}$

$= \frac{k}{6} \times \sqrt{\frac{6}{k}} = \sqrt{\frac{k}{6}}$

$\therefore$ maximum V when $l = x\ \therefore$ prism is a cube

1 **a** $f'(x) = 6x^2 + 10x$

b $6x^2 + 10x \geq 0$
$2x(3x + 5) \geq 0$
$x \leq -\frac{5}{3}$ and $x \geq 0$

2 **a** $\frac{dy}{dx} = 3x^2 - 2x + 2$
at $(1, -2)$, grad = 3
$\therefore\ y + 2 = 3(x - 1)$
$3x - y - 5 = 0$

b SP when $3x^2 - 2x + 2 = 0$
$b^2 - 4ac = 4 - 24 = -20$
$b^2 - 4ac < 0\ \therefore$ no real roots
$\therefore$ no stationary points

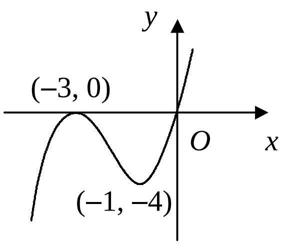
3 **a** $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - 4x^{-2}$
$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}} + 8x^{-3}$

b SP: $\frac{1}{2}x^{-\frac{1}{2}} - 4x^{-2} = 0$
$\frac{1}{2}x^{-2}(x^{\frac{3}{2}} - 8) = 0$
$x^{\frac{3}{2}} = 8$
$x = 4$
$\therefore$ (4, 3)
when $x = 4$, $\frac{d^2y}{dx^2} = \frac{3}{32}$
$\frac{d^2y}{dx^2} > 0\ \therefore$ minimum

4 **a** $y = 0 \Rightarrow x(x + 3)^2 = 0$
$x = -3, 0$
$\therefore$ (–3, 0), (0, 0)

b $f'(x) = 3x^2 + 12x + 9$
decreasing when $3x^2 + 12x + 9 \leq 0$
$3(x + 3)(x + 1) \leq 0$
$\therefore\ -3 \leq x \leq -1$

c



5 **a** $\frac{dh}{dt} = 8t^3 - 24t^2 + 16t$

b when $t = 0.25$,
$\frac{dh}{dt} = 2.625$ cm per second

c SP: $8t^3 - 24t^2 + 16t = 0$
$8t(t - 1)(t - 2) = 0$
$t = 0, 1, 2$
from graph, max when $t = 1$
$\therefore$ max height = 3 cm

6 **a** $\frac{dy}{dx} = 3x^2 + 6kx - 9k^2$
stationary when $3x^2 + 6kx - 9k^2 = 0$
$\Rightarrow\ x^2 + 2kx - 3k^2 = 0$

b $(x + 3k)(x - k) = 0$
$x = -3k, k$
when $x = k$, $y = k^3 + 3k^3 - 9k^3 = -5k^3$
$\therefore$ stationary at $(k, -5k^3)$

c when $x = -3k$,
$y = -27k^3 + 27k^3 + 27k^3 = 27k^3$
$\therefore\ (-3k, 27k^3)$

7 **a** $V = \frac{1}{2}x^2 \sin 60° \times l$

$= \frac{1}{2}x^2 l \times \frac{\sqrt{3}}{2} = 250$

$\therefore\ l = \frac{1000}{\sqrt{3}x^2}$ or $\frac{1000\sqrt{3}}{3x^2}$

b $A = (2 \times \frac{\sqrt{3}}{4}x^2) + 3xl$

$= \frac{\sqrt{3}}{2}x^2 + (3x \times \frac{1000\sqrt{3}}{3x^2})$

$= \frac{\sqrt{3}}{2}(x^2 + \frac{2000}{x})$

c $\frac{dA}{dx} = \frac{\sqrt{3}}{2}(2x - 2000x^{-2})$

SP: $\frac{\sqrt{3}}{2}(2x - 2000x^{-2}) = 0$

$x^3 = 1000$

$x = 10$

d min $A = 150\sqrt{3}$

e $\frac{d^2A}{dx^2} = \frac{\sqrt{3}}{2}(2 + 4000x^{-3})$

when $x = 10$, $\frac{d^2A}{dx^2} = 3\sqrt{3}$

$\frac{d^2A}{dx^2} > 0\ \therefore$ minimum

8 **a** $f'(x) = 3x^2 + 8x + k$

for 2 SPs, $f'(x) = 0$ has 2 distinct roots

$\therefore\ b^2 - 4ac > 0$

$64 - 12k > 0$

$k < \frac{16}{3}$

b SP: $3x^2 + 8x - 3 = 0$

$(3x - 1)(x + 3) = 0$

$x = -3, \frac{1}{3}$

$\therefore\ (-3, 19)$ and $(\frac{1}{3}, \frac{13}{27})$

9 **a** $x^{\frac{1}{2}} - 4 + 3x^{-\frac{1}{2}} = 0$

$x - 4x^{\frac{1}{2}} + 3 = 0$

$(x^{\frac{1}{2}} - 1)(x^{\frac{1}{2}} - 3) = 0$

$x^{\frac{1}{2}} = 1, 3$

$x = 1, 9$

$\therefore$ (1, 0) and (9, 0)

b $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

SP: $\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} = 0$

$\frac{1}{2}x^{-\frac{3}{2}}(x - 3) = 0$

$x = 3$

$y = \sqrt{3} - 4 + \frac{3}{\sqrt{3}} = 2\sqrt{3} - 4$

$\therefore\ (3, 2\sqrt{3} - 4)$

10 **a** $f(-1) = -1 - 3 + 4 = 0$

$\therefore\ (x + 1)$ is a factor

b

$$\begin{array}{r|l} & x^2 - 4x + 4 \\ \hline x+1 & x^3 - 3x^2 + 0x + 4 \\ & x^3 + x^2 \\ \hline & -4x^2 + 0x \\ & -4x^2 - 4x \\ \hline & 4x + 4 \\ & 4x + 4 \\ \hline \end{array}$$

$\therefore\ f(x) \equiv (x + 1)(x^2 - 4x + 4)$

$f(x) \equiv (x + 1)(x - 2)^2$

c (2, 0), as $(x - 2)$ is a repeated factor of f(x) so x-axis is a tangent at (2, 0)

d $f'(x) = 3x^2 - 6x$

SP: $3x^2 - 6x = 0$

$3x(x - 2) = 0$

$x = 0, 2$

$\therefore$ (0, 4) is other turning point

1 **a** $f'(x) = 24 + 6x - 3x^2$

b $24 + 6x - 3x^2 \geq 0$

$x^2 - 2x - 8 \leq 0$

$(x + 2)(x - 4) \leq 0$

$-2 \leq x \leq 4$

2 **a** $(-2, 30) \Rightarrow 30 = -8 + 4a + 48 + b$

$\therefore\ 4a + b + 10 = 0$

b $\frac{dy}{dx} = 3x^2 + 2ax - 24$

SP at P $\therefore\ \frac{dy}{dx} = 0$

$\Rightarrow\ 12 - 4a - 24 = 0$

$a = -3,\ b = 2$

c $3x^2 - 6x - 24 = 0$

$3(x + 2)(x - 4) = 0$

$x = -2$ (at P) or 4

other SP $(4, -78)$

3 **a** $f'(x) = 2x - 16x^{-2}$

b SP: $2x - 16x^{-2} = 0$

$x^3 = 8$

$x = 2$

$\therefore\ (2, 12)$

$f''(x) = 2 + 32x^{-3}$

$f''(2) = 6$

$f''(x) > 0\ \therefore$ minimum

4 **a** area $= (2 \times \frac{1}{2}r^2\theta) + \frac{1}{2}r^2(3\theta) = 25$

$\therefore\ \frac{5}{2}r^2\theta = 25,\ \theta = \frac{10}{r^2}$

b $P = 2r + (2 \times r\theta) + r(3\theta) = 2r + 5r\theta$

$= 2r + 5r(\frac{10}{r^2}) = 2r + \frac{50}{r}$

c $\frac{dP}{dr} = 2 - 50r^{-2}$

SP: $2 - 50r^{-2} = 0$

$r^2 = 25$

$r = 5$

d min $P = 20$

e $\frac{d^2P}{dr^2} = 100r^{-3}$, when $r = 5$, $\frac{d^2P}{dr^2} = 0.8$

$\frac{d^2P}{dr^2} > 0\ \therefore$ minimum

5 **a** $2x - x^{\frac{3}{2}} = 0$

$x(2 - x^{\frac{1}{2}}) = 0$

$x = 0$ or $x^{\frac{1}{2}} = 2 \Rightarrow x = 4$

$\therefore\ (0, 0)$ and $(4, 0)$

b $\frac{dy}{dx} = 2 - \frac{3}{2}x^{\frac{1}{2}}$

SP: $2 - \frac{3}{2}x^{\frac{1}{2}} = 0$

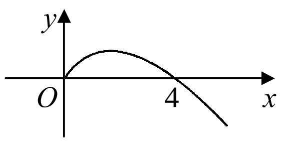
$x^{\frac{1}{2}} = \frac{4}{3}$

$x = \frac{16}{9}$

$\frac{d^2y}{dx^2} = -\frac{3}{4}x^{-\frac{1}{2}}$, when $x = \frac{16}{9}$, $\frac{d^2y}{dx^2} = -\frac{9}{16}$

$\frac{d^2y}{dx^2} < 0\ \therefore$ maximum

c



6 **a** $\frac{dy}{dx} = 3x^2 - 3$

SP: $3x^2 - 3 = 0$

$x^2 = 1$

$x = \pm 1$

$\therefore\ (-1, 3)$ and $(1, -1)$

b $PQ^2 = 2^2 + 4^2 = 20$

$\therefore\ PQ = \sqrt{20} = 2\sqrt{5}$

7 **a** $2x - 5 + \frac{2}{x} = 0$

$2x^2 - 5x + 2 = 0$

$(2x - 1)(x - 2) = 0$

$x = \frac{1}{2}, 2$

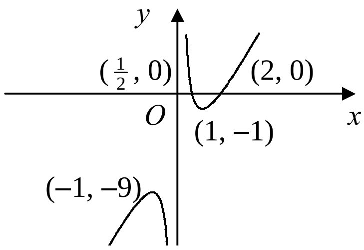
b $f'(x) = 2 - 2x^{-2}$

$\therefore\ 2 - 2x^{-2} = 0$

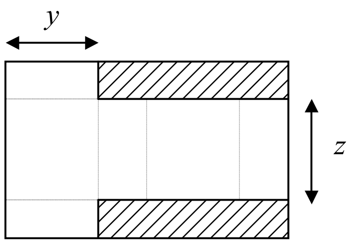
$x^2 = 1$

$x = \pm 1$

c



8 **a**



$2x + z = 25$

$2x + 2y = 40$

$\therefore$ length and width $(25 - 2x)$ and $(20 - x)$

b volume $= x(25 - 2x)(20 - x)$

$= x(500 - 65x + 2x^2)$

$= 2x^3 - 65x^2 + 500x$

c $\frac{dV}{dx} = 6x^2 - 130x + 500$

SP: $6x^2 - 130x + 500 = 0$

$2(3x - 50)(x - 5) = 0$

$x = 5, \frac{50}{3}$

$2x < 25 \ \therefore\ x < 12.5$

$\therefore\ x = 5$

d max volume = 1125 cm^3

$\frac{d^2V}{dx^2} = 12x - 130$

when $x = 5$, $\frac{d^2V}{dx^2} = -70$

$\frac{d^2V}{dx^2} < 0 \ \therefore$ maximum

9 **a** $\frac{dy}{dx} = 9 + 6x - 3x^2$

SP: $9 + 6x - 3x^2 = 0$

$-3(x + 1)(x - 3) = 0$

$x = -1, 3$

$\therefore\ (-1, -3)$ and $(3, 29)$

b $\frac{d^2y}{dx^2} = 6 - 6x$

$(-1, -3)$: $\frac{d^2y}{dx^2} = 12 \ \therefore$ minimum

$(3, 29)$: $\frac{d^2y}{dx^2} = -12 \ \therefore$ maximum

c $-3 < k < 29$

10 **a** $f(-1) = 15$

$\therefore\ -4 + a + 12 + b = 15$

$a + b = 7$ (1)

b $f(2) = 42$

$\therefore\ 32 + 4a - 24 + b = 42$

$4a + b = 34$ (2)

(2) – (1) $3a = 27$

$\therefore\ a = 9,\ b = -2$

c $f(x) = 4x^3 + 9x^2 - 12x - 2$

$f'(x) = 12x^2 + 18x - 12$

SP: $12x^2 + 18x - 12 = 0$

$2x^2 + 3x - 2 = 0$

$(2x - 1)(x + 2) = 0$

$x = -2, \frac{1}{2}$

$\therefore\ (-2, 26)$ and $(\frac{1}{2}, -\frac{21}{4})$