C4 SERIES Worksheet A

1 Find the binomial expansion of each of the following in ascending powers of x up to and including the term in x^3, for $|x| < 1$.

a $(1 + x)^{-1}$ **b** $(1 + x)^{\frac{1}{2}}$ **c** $2(1 + x)^{-3}$ **d** $(1 + x)^{\frac{2}{3}}$

e $\sqrt[3]{1-x}$ **f** $\frac{1}{(1+x)^2}$ **g** $\frac{1}{4(1-x)^4}$ **h** $\frac{3}{\sqrt{1-x}}$

2 Expand each of the following in ascending powers of x up to and including the term in x^3 and state the set of values of x for which each expansion is valid.

a $(1 + 2x)^{\frac{1}{2}}$ **b** $(1 - 3x)^{-1}$ **c** $(1 - 4x)^{-\frac{1}{2}}$ **d** $(1 + \frac{1}{2}x)^{-3}$

e $(1 - 6x)^{\frac{1}{3}}$ **f** $(1 + \frac{1}{4}x)^{-4}$ **g** $(1 + 2x)^{\frac{3}{2}}$ **h** $(1 - 3x)^{-\frac{4}{3}}$

3 **a** Expand $(1 - 2x)^{\frac{1}{2}}$, $|x| < \frac{1}{2}$, in ascending powers of x up to and including the term in x^3.

b By substituting a suitable value of x in your expansion, find an estimate for $\sqrt{0.98}$

c Show that $\sqrt{0.98} = \frac{7}{10}\sqrt{2}$ and hence find the value of $\sqrt{2}$ correct to 8 significant figures.

4 Expand each of the following in ascending powers of x up to and including the term in x^3 and state the set of values of x for which each expansion is valid.

a $(2 + x)^{-1}$ **b** $(4 + x)^{\frac{1}{2}}$ **c** $(3 - x)^{-3}$ **d** $(9 + 3x)^{\frac{1}{2}}$

e $(8 - 24x)^{\frac{1}{3}}$ **f** $(4 - 3x)^{-1}$ **g** $(4 + 6x)^{-\frac{1}{2}}$ **h** $(3 + 2x)^{-2}$

5 **a** Expand $(1 + 2x)^{-1}$, $|x| < \frac{1}{2}$, in ascending powers of x up to and including the term in x^3.

b Hence find the series expansion of $\frac{1-x}{1+2x}$, $|x| < \frac{1}{2}$, in ascending powers of x up to and including the term in x^3.

6 Find the first four terms in the series expansion in ascending powers of x of each of the following and state the set of values of x for which each expansion is valid.

a $\frac{1+3x}{1-x}$ **b** $\frac{2x-1}{(1+4x)^2}$ **c** $\frac{3+x}{2-x}$ **d** $\frac{1-x}{\sqrt{1+2x}}$

7 **a** Express $\frac{x-2}{(1-x)(1-2x)}$ in partial fractions.

b Hence find the series expansion of $\frac{x-2}{(1-x)(1-2x)}$ in ascending powers of x up to and including the term in x^3 and state the set of values of x for which the expansion is valid.

8 By first expressing f(x) in partial fractions, find the series expansion of f(x) in ascending powers of x up to and including the term in x^3 and state the set of values of x for which it is valid.

a $f(x) \equiv \frac{4}{(1+x)(1-3x)}$ **b** $f(x) \equiv \frac{1-6x}{1+3x-4x^2}$ **c** $f(x) \equiv \frac{5}{2-3x-2x^2}$

d $f(x) \equiv \frac{7x-3}{x^2-4x+3}$ **e** $f(x) \equiv \frac{3+5x}{(1+3x)(1+x)^2}$ **f** $f(x) \equiv \frac{2x^2+4}{2x^2+x-1}$

C4 SERIES Worksheet B

1 **a** Expand $(1 - x)^{\frac{1}{2}}$, $|x| < 1$, in ascending powers of x up to and including the term in x^3.

b By substituting $x = 0.01$ in your expansion, find the value of $\sqrt{11}$ correct to 9 significant figures.

2 The series expansion of $(1 + 8x)^{\frac{1}{2}}$, in ascending powers of x up to and including the term in x^3, is

$$1 + 4x + ax^2 + bx^3, \quad |x| < \tfrac{1}{8}.$$

a Find the values of the constants a and b.

b Use the expansion, with $x = 0.01$, to find the value of $\sqrt{3}$ to 5 decimal places.

3 **a** Expand $(9 - 6x)^{\frac{1}{2}}$, $|x| < \frac{3}{2}$, in ascending powers of x up to and including the term in x^3, simplifying the coefficient in each term.

b Use your expansion with a suitable value of x to find the value of $\sqrt{8.7}$ correct to 7 significant figures.

4 **a** Expand $(1 + 6x)^{\frac{1}{3}}$, $|x| < \frac{1}{6}$, in ascending powers of x up to and including the term in x^3.

b Use your expansion, with $x = 0.004$, to find the cube root of 2 correct to 7 significant figures.

5 **a** Expand $(1 + 2x)^{-3}$ in ascending powers of x up to and including the term in x^3 and state the set of values of x for which the expansion is valid.

b Hence, or otherwise, find the series expansion in ascending powers of x up to and including the term in x^3 of $\dfrac{1+3x}{(1+2x)^3}$.

6 Find the coefficient of x^2 in the series expansion of $\dfrac{2+x}{\sqrt{4-2x}}$, $|x| < 2$.

7 **a** Find the values of A and B such that

$$\frac{2-11x}{1-5x+4x^2} \equiv \frac{A}{1-x} + \frac{B}{1-4x}.$$

b Hence, find the series expansion of $\dfrac{2-11x}{1-5x+4x^2}$ in ascending powers of x up to and including the term in x^3 and state the set of values of x for which the expansion is valid.

8
$$\text{f}(x) \equiv \frac{4-17x}{(1+2x)(1-3x)^2}, \quad |x| < \tfrac{1}{3}.$$

a Express f(x) in partial fractions.

b Hence, or otherwise, find the series expansion of f(x) in ascending powers of x up to and including the term in x^3.

9 The first three terms in the expansion of $(1 + ax)^b$, in ascending powers of x, for $|ax| < 1$, are

$$1 - 6x + 24x^2.$$

a Find the values of the constants a and b.

b Find the coefficient of x^3 in the expansion.

Back to Previous Menu

C4 SERIES — Worksheet C

1 **a** Expand $(1-4x)^{\frac{1}{2}}$ in ascending powers of x up to and including the term in x^3 and state the set of values of x for which the expansion is valid. **(4)**

b By substituting $x = 0.01$ in your expansion, find the value of $\sqrt{6}$ to 6 significant figures. **(3)**

2
$$\text{f}(x) \equiv \frac{4}{1+2x-3x^2}.$$

a Express f(x) in partial fractions. **(3)**

b Hence, or otherwise, find the series expansion of f(x) in ascending powers of x up to and including the term in x^3 and state the set of values of x for which the expansion is valid. **(5)**

3 **a** Expand $(2-x)^{-2}$, $|x| < 2$, in ascending powers of x up to and including the term in x^3. **(4)**

b Hence, find the coefficient of x^3 in the series expansion of $\dfrac{3-x}{(2-x)^2}$. **(2)**

4
$$\text{f}(x) \equiv \frac{4}{\sqrt{1+\frac{2}{3}x}}, \quad -\tfrac{3}{2} < x < \tfrac{3}{2}.$$

a Show that $\text{f}(\frac{1}{10}) = \sqrt{15}$. **(2)**

b Expand f(x) in ascending powers of x up to and including the term in x^2. **(3)**

c Use your expansion to obtain an approximation for $\sqrt{15}$, giving your answer as an exact, simplified fraction. **(2)**

d Show that $3\frac{55}{63}$ is a more accurate approximation for $\sqrt{15}$. **(2)**

5 **a** Expand $(1-x)^{\frac{1}{3}}$, $|x| < 1$, in ascending powers of x up to and including the term in x^2. **(3)**

b By substituting $x = 10^{-3}$ in your expansion, find the cube root of 37 correct to 9 significant figures. **(3)**

6 The series expansion of $(1+5x)^{\frac{3}{5}}$, in ascending powers of x up to and including the term in x^3, is

$$1 + 3x + px^2 + qx^3, \quad |x| < \tfrac{1}{5}.$$

a Find the values of the constants p and q. **(4)**

b Use the expansion with a suitable value of x to find an approximate value for $(1.1)^{\frac{3}{5}}$. **(2)**

c Obtain the value of $(1.1)^{\frac{3}{5}}$ from your calculator and hence find the percentage error in your answer to part **b**. **(2)**

7 **a** Find the values of A, B and C such that

$$\frac{8-6x^2}{(1+x)(2+x)^2} \equiv \frac{A}{1+x} + \frac{B}{2+x} + \frac{C}{(2+x)^2}.$$ **(4)**

b Hence find the series expansion of $\dfrac{8-6x^2}{(1+x)(2+x)^2}$, $|x| < 1$, in ascending powers of x up to and including the term in x^3, simplifying each coefficient. **(7)**

Back to Previous Menu

8 **a** Expand $(1 - 2x)^{\frac{1}{2}}$, $|x| < \frac{1}{2}$, in ascending powers of x up to and including the term in x^2. **(3)**

b By substituting $x = 0.0008$ in your expansion, find the square root of 39 correct to 7 significant figures. **(4)**

9 **a** Find the series expansion of $(1 + 8x)^{\frac{1}{3}}$, $|x| < \frac{1}{8}$, in ascending powers of x up to and including the term in x^2, simplifying each term. **(3)**

b Find the exact fraction k such that

$$\sqrt[3]{5} = k\sqrt[3]{1.08}$$ **(2)**

c Hence, use your answer to part **a** together with a suitable value of x to obtain an estimate for $\sqrt[3]{5}$, giving your answer to 4 significant figures. **(3)**

10
$$\text{f}(x) \equiv \frac{6x}{x^2 - 4x + 3}, \quad |x| < 1.$$

a Express f(x) in partial fractions. **(3)**

b Show that for small values of x,

$$\text{f}(x) \approx 2x + \tfrac{8}{3}x^2 + \tfrac{26}{9}x^3.$$ **(5)**

11 **a** Find the binomial expansion of $(4 + x)^{\frac{1}{2}}$ in ascending powers of x up to and including the term in x^2 and state the set of values of x for which the expansion is valid. **(4)**

b By substituting $x = \frac{1}{20}$ in your expansion, find an estimate for $\sqrt{5}$, giving your answer to 9 significant figures. **(3)**

c Obtain the value of $\sqrt{5}$ from your calculator and hence comment on the accuracy of the estimate found in part **b**. **(2)**

12 **a** Expand $(1 + 2x)^{-\frac{1}{2}}$, $|x| < \frac{1}{2}$, in ascending powers of x up to and including the term in x^3. **(4)**

b Hence, show that for small values of x,

$$\frac{2 - 5x}{\sqrt{1 + 2x}} \approx 2 - 7x + 8x^2 - \tfrac{25}{2}x^3.$$ **(3)**

c Solve the equation

$$\frac{2 - 5x}{\sqrt{1 + 2x}} = \sqrt{3}.$$ **(3)**

d Use your answers to parts **b** and **c** to find an approximate value for $\sqrt{3}$. **(2)**

13 **a** Expand $(1 + x)^{-1}$, $|x| < 1$, in ascending powers of x up to and including the term in x^3. **(2)**

b Hence, write down the first four terms in the expansion in ascending powers of x of $(1 + bx)^{-1}$, where b is a constant, for $|bx| < 1$. **(1)**

Given that in the series expansion of

$$\frac{1 + ax}{1 + bx}, \quad |bx| < 1,$$

the coefficient of x is -4 and the coefficient of x^2 is 12,

c find the values of the constants a and b, **(5)**

d find the coefficient of x^3 in the expansion. **(2)**

C4 SERIES — Answers - Worksheet A

1 **a** $= 1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3\times2}x^3 + \ldots$
$= 1 - x + x^2 - x^3 + \ldots$

b $= 1 + (\frac{1}{2})x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}x^3 + \ldots$
$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$

c $= 2[1 + (-3)x + \frac{(-3)(-4)}{2}x^2 + \frac{(-3)(-4)(-5)}{3\times2}x^3 + \ldots]$
$= 2 - 6x + 12x^2 - 20x^3 + \ldots$

d $= 1 + (\frac{2}{3})x + \frac{(\frac{2}{3})(-\frac{1}{3})}{2}x^2 + \frac{(\frac{2}{3})(-\frac{1}{3})(-\frac{4}{3})}{3\times2}x^3 + \ldots$
$= 1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 + \ldots$

e $= (1 - x)^{\frac{1}{3}} = 1 + (\frac{1}{3})(-x) + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}(-x)^2 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3\times2}(-x)^3 + \ldots$
$= 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 + \ldots$

f $= (1 + x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2}x^2 + \frac{(-2)(-3)(-4)}{3\times2}x^3 + \ldots$
$= 1 - 2x + 3x^2 - 4x^3 + \ldots$

g $= \frac{1}{4}(1 - x)^{-4} = \frac{1}{4}[1 + (-4)(-x) + \frac{(-4)(-5)}{2}(-x)^2 + \frac{(-4)(-5)(-6)}{3\times2}(-x)^3 + \ldots]$
$= \frac{1}{4} + x + \frac{5}{2}x^2 + 5x^3 + \ldots$

h $= 3(1 - x)^{-\frac{1}{2}} = 3[1 + (-\frac{1}{2})(-x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3\times2}(-x)^3 + \ldots]$
$= 3 + \frac{3}{2}x + \frac{9}{8}x^2 + \frac{15}{16}x^3 + \ldots$

2 **a** $= 1 + (\frac{1}{2})(2x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(2x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}(2x)^3 + \ldots$
$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \ldots,\ |2x| < 1 \quad \therefore$ valid for $|x| < \frac{1}{2}$

b $= 1 + (-1)(-3x) + \frac{(-1)(-2)}{2}(-3x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(-3x)^3 + \ldots$
$= 1 + 3x + 9x^2 + 27x^3 + \ldots,\ |-3x| < 1 \quad \therefore$ valid for $|x| < \frac{1}{3}$

c $= 1 + (-\frac{1}{2})(-4x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-4x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3\times2}(-4x)^3 + \ldots$
$= 1 + 2x + 6x^2 + 20x^3 + \ldots,\ |-4x| < 1 \quad \therefore$ valid for $|x| < \frac{1}{4}$

d $= 1 + (-3)(\frac{1}{2}x) + \frac{(-3)(-4)}{2}(\frac{1}{2}x)^2 + \frac{(-3)(-4)(-5)}{3\times2}(\frac{1}{2}x)^3 + \ldots$
$= 1 - \frac{3}{2}x + \frac{3}{2}x^2 - \frac{5}{4}x^3 + \ldots,\ |\frac{1}{2}x| < 1 \quad \therefore$ valid for $|x| < 2$

e $= 1 + (\frac{1}{3})(-6x) + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}(-6x)^2 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3\times2}(-6x)^3 + \ldots$
$= 1 - 2x - 4x^2 - \frac{40}{3}x^3 + \ldots,\ |-6x| < 1 \quad \therefore$ valid for $|x| < \frac{1}{6}$

f $= 1 + (-4)(\frac{1}{4}x) + \frac{(-4)(-5)}{2}(\frac{1}{4}x)^2 + \frac{(-4)(-5)(-6)}{3\times2}(\frac{1}{4}x)^3 + \ldots$
$= 1 - x + \frac{5}{8}x^2 - \frac{5}{16}x^3 + \ldots,\ |\frac{1}{4}x| < 1 \quad \therefore$ valid for $|x| < 4$

g $= 1 + (\frac{3}{2})(2x) + \frac{(\frac{3}{2})(\frac{1}{2})}{2}(2x)^2 + \frac{(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})}{3\times2}(2x)^3 + \ldots$
$= 1 + 3x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \ldots,\ |2x| < 1 \quad \therefore$ valid for $|x| < \frac{1}{2}$

h $= 1 + (-\frac{4}{3})(-3x) + \frac{(-\frac{4}{3})(-\frac{7}{3})}{2}(-3x)^2 + \frac{(-\frac{4}{3})(-\frac{7}{3})(-\frac{10}{3})}{3\times2}(-3x)^3 + \ldots$
$= 1 + 4x + 14x^2 + \frac{140}{3}x^3 + \ldots,\ |-3x| < 1 \quad \therefore$ valid for $|x| < \frac{1}{3}$

Back to Previous Menu

3 **a** $= 1 + (\tfrac{1}{2})(-2x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-2x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}(-2x)^3 + \ldots$

$= 1 - x - \tfrac{1}{2}x^2 - \tfrac{1}{2}x^3 + \ldots$

b $\sqrt{0.98} = (1 - 2x)^{\frac{1}{2}}$ when $x = 0.01$

$\therefore \sqrt{0.98} \approx 1 - (0.01) - \tfrac{1}{2}(0.01)^2 - \tfrac{1}{2}(0.01)^3$

$= 1 - 0.01 - 0.000\,05 - 0.000\,000\,5$

$= 0.989\,949\,5$

c $\sqrt{0.98} = \sqrt{\frac{98}{100}} = \sqrt{\frac{49\times2}{100}} = \frac{7}{10}\sqrt{2}$

$\therefore \sqrt{2} \approx \frac{10}{7} \times 0.989\,949\,5 = 1.414\,213\,6$ (8sf)

4 **a** $= 2^{-1}(1 + \tfrac{1}{2}x)^{-1} = \tfrac{1}{2}(1 + \tfrac{1}{2}x)^{-1}$

$= \tfrac{1}{2}[1 + (-1)(\tfrac{1}{2}x) + \frac{(-1)(-2)}{2}(\tfrac{1}{2}x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(\tfrac{1}{2}x)^3 + \ldots]$

$= \tfrac{1}{2} - \tfrac{1}{4}x + \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 + \ldots,\ |\tfrac{1}{2}x| < 1 \quad \therefore$ valid for $|x| < 2$

b $= 4^{\frac{1}{2}}(1+\tfrac{1}{4}x)^{\frac{1}{2}} = 2(1 + \tfrac{1}{4}x)^{\frac{1}{2}}$

$= 2[1 + (\tfrac{1}{2})(\tfrac{1}{4}x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(\tfrac{1}{4}x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}(\tfrac{1}{4}x)^3 + \ldots]$

$= 2 + \tfrac{1}{4}x - \tfrac{1}{64}x^2 + \tfrac{1}{512}x^3 + \ldots,\ |\tfrac{1}{4}x| < 1 \quad \therefore$ valid for $|x| < 4$

c $= 3^{-3}(1 - \tfrac{1}{3}x)^{-3} = \tfrac{1}{27}(1 - \tfrac{1}{3}x)^{-3}$

$= \tfrac{1}{27}[1 + (-3)(-\tfrac{1}{3}x) + \frac{(-3)(-4)}{2}(-\tfrac{1}{3}x)^2 + \frac{(-3)(-4)(-5)}{3\times2}(-\tfrac{1}{3}x)^3 + \ldots]$

$= \tfrac{1}{27} + \tfrac{1}{27}x + \tfrac{2}{81}x^2 + \tfrac{10}{729}x^3 + \ldots,\ |-\tfrac{1}{3}x| < 1 \quad \therefore$ valid for $|x| < 3$

d $= 9^{\frac{1}{2}}(1 + \tfrac{1}{3}x)^{\frac{1}{2}} = 3(1 + \tfrac{1}{3}x)^{\frac{1}{2}}$

$= 3[1 + (\tfrac{1}{2})(\tfrac{1}{3}x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(\tfrac{1}{3}x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}(\tfrac{1}{3}x)^3 + \ldots]$

$= 3 + \tfrac{1}{2}x - \tfrac{1}{24}x^2 + \tfrac{1}{144}x^3 + \ldots,\ |\tfrac{1}{3}x| < 1 \quad \therefore$ valid for $|x| < 3$

e $= 8^{\frac{1}{3}}(1 - 3x)^{\frac{1}{3}} = 2(1 - 3x)^{\frac{1}{3}}$

$= 2[1 + (\tfrac{1}{3})(-3x) + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}(-3x)^2 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3\times2}(-3x)^3 + \ldots]$

$= 2 - 2x - 2x^2 - \tfrac{10}{3}x^3 + \ldots,\ |-3x| < 1 \quad \therefore$ valid for $|x| < \tfrac{1}{3}$

f $= 4^{-1}(1 - \tfrac{3}{4}x)^{-1} = \tfrac{1}{4}(1 - \tfrac{3}{4}x)^{-1}$

$= \tfrac{1}{4}[1 + (-1)(-\tfrac{3}{4}x) + \frac{(-1)(-2)}{2}(-\tfrac{3}{4}x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(-\tfrac{3}{4}x)^3 + \ldots]$

$= \tfrac{1}{4} + \tfrac{3}{16}x + \tfrac{9}{64}x^2 + \tfrac{27}{256}x^3 + \ldots,\ |-\tfrac{3}{4}x| < 1 \quad \therefore$ valid for $|x| < \tfrac{4}{3}$

g $= 4^{-\frac{1}{2}}(1 + \tfrac{3}{2}x)^{-\frac{1}{2}} = \tfrac{1}{2}(1 + \tfrac{3}{2}x)^{-\frac{1}{2}}$

$= \tfrac{1}{2}[1 + (-\tfrac{1}{2})(\tfrac{3}{2}x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\tfrac{3}{2}x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3\times2}(\tfrac{3}{2}x)^3 + \ldots]$

$= \tfrac{1}{2} - \tfrac{3}{8}x + \tfrac{27}{64}x^2 - \tfrac{135}{256}x^3 + \ldots,\ |\tfrac{3}{2}x| < 1 \quad \therefore$ valid for $|x| < \tfrac{2}{3}$

h $= 3^{-2}(1 + \tfrac{2}{3}x)^{-2} = \tfrac{1}{9}(1 + \tfrac{2}{3}x)^{-2}$

$= \tfrac{1}{9}[1 + (-2)(\tfrac{2}{3}x) + \frac{(-2)(-3)}{2}(\tfrac{2}{3}x)^2 + \frac{(-2)(-3)(-4)}{3\times2}(\tfrac{2}{3}x)^3 + \ldots]$

$= \tfrac{1}{9} - \tfrac{4}{27}x + \tfrac{4}{27}x^2 - \tfrac{32}{243}x^3 + \ldots,\ |\tfrac{2}{3}x| < 1 \quad \therefore$ valid for $|x| < \tfrac{3}{2}$

Back to Previous Menu

5 **a** $= 1 + (-1)(2x) + \frac{(-1)(-2)}{2}(2x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(2x)^3 + \ldots$

$= 1 - 2x + 4x^2 - 8x^3 + \ldots$

b $= (1 - x)(1 + 2x)^{-1} = (1 - x)(1 - 2x + 4x^2 - 8x^3 + \ldots)$

$= 1 - 2x + 4x^2 - 8x^3 - x + 2x^2 - 4x^3 + \ldots$

$= 1 - 3x + 6x^2 - 12x^3 + \ldots$

6 **a** $= (1 + 3x)(1 - x)^{-1} = (1 + 3x)[1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-x)^3 + \ldots]$

$= (1 + 3x)(1 + x + x^2 + x^3 + \ldots)$

$= 1 + x + x^2 + x^3 + 3x + 3x^2 + 3x^3 + \ldots$

$= 1 + 4x + 4x^2 + 4x^3 + \ldots, \ |-x| < 1 \qquad \therefore$ valid for $|x| < 1$

b $= (2x - 1)(1 + 4x)^{-2} = (2x - 1)[1 + (-2)(4x) + \frac{(-2)(-3)}{2}(4x)^2 + \frac{(-2)(-3)(-4)}{3\times 2}(4x)^3 + \ldots]$

$= (2x - 1)(1 - 8x + 48x^2 - 256x^3 + \ldots)$

$= 2x - 16x^2 + 96x^3 - 1 + 8x - 48x^2 + 256x^3 + \ldots$

$= -1 + 10x - 64x^2 + 352x^3 + \ldots, \ |4x| < 1 \quad \therefore$ valid for $|x| < \frac{1}{4}$

c $= (3 + x)(2 - x)^{-1} = (3 + x) \times 2^{-1}(1 - \frac{1}{2}x)^{-1}$

$= (3 + x) \times \frac{1}{2}[1 + (-1)(-\frac{1}{2}x) + \frac{(-1)(-2)}{2}(-\frac{1}{2}x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-\frac{1}{2}x)^3 + \ldots]$

$= (3 + x)(\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots)$

$= \frac{3}{2} + \frac{3}{4}x + \frac{3}{8}x^2 + \frac{3}{16}x^3 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \ldots$

$= \frac{3}{2} + \frac{5}{4}x + \frac{5}{8}x^2 + \frac{5}{16}x^3 + \ldots, \ |-\frac{1}{2}x| < 1 \qquad \therefore$ valid for $|x| < 2$

d $= (1 - x)(1 + 2x)^{-\frac{1}{2}} = (1 - x)[1 + (-\frac{1}{2})(2x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(2x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3\times 2}(2x)^3 + \ldots]$

$= (1 - x)(1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \ldots)$

$= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 - x + x^2 - \frac{3}{2}x^3 + \ldots$

$= 1 - 2x + \frac{5}{2}x^2 - 4x^3 + \ldots, \ |2x| < 1 \qquad \therefore$ valid for $|x| < \frac{1}{2}$

7 **a** $\frac{x-2}{(1-x)(1-2x)} \equiv \frac{A}{1-x} + \frac{B}{1-2x}$

$x - 2 \equiv A(1 - 2x) + B(1 - x)$

$x = 1 \quad \Rightarrow \quad -1 = -A \quad \Rightarrow \quad A = 1$

$x = \frac{1}{2} \quad \Rightarrow \quad -\frac{3}{2} = \frac{1}{2}B \quad \Rightarrow \quad B = -3$

$\therefore \ \frac{x-2}{(1-x)(1-2x)} \equiv \frac{1}{1-x} - \frac{3}{1-2x}$

b $\frac{1}{1-x} = (1 - x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-x)^3 + \ldots$

$= 1 + x + x^2 + x^3 + \ldots, \ |-x| < 1 \ \therefore |x| < 1$

$\frac{3}{1-2x} = 3(1 - 2x)^{-1} = 3[1 + (-1)(-2x) + \frac{(-1)(-2)}{2}(-2x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-2x)^3 + \ldots]$

$= 3 + 6x + 12x^2 + 24x^3 + \ldots, \ |-2x| < 1 \ \therefore \ |x| < \frac{1}{2}$

$\therefore \ \frac{x-2}{(1-x)(1-2x)} = (1 + x + x^2 + x^3 + \ldots) - (3 + 6x + 12x^2 + 24x^3 + \ldots)$

$= -2 - 5x - 11x^2 - 23x^3 + \ldots,$ valid for $|x| < \frac{1}{2}$

Back to Previous Menu

8 **a** $\frac{4}{(1+x)(1-3x)} \equiv \frac{A}{1+x} + \frac{B}{1-3x}$

$4 \equiv A(1-3x) + B(1+x)$

$x = -1 \quad \Rightarrow \quad 4 = 4A \quad \Rightarrow \quad A = 1$

$x = \frac{1}{3} \quad \Rightarrow \quad 4 = \frac{4}{3}B \quad \Rightarrow \quad B = 3$

$\therefore \; \text{f}(x) \equiv \frac{1}{1+x} + \frac{3}{1-3x}$

$\frac{1}{1+x} = (1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3\times2}x^3 + \ldots$

$= 1 - x + x^2 - x^3 + \ldots, \; |x| < 1$

$\frac{3}{1-3x} = 3(1-3x)^{-1} = 3[1 + (-1)(-3x) + \frac{(-1)(-2)}{2}(-3x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(-3x)^3 + \ldots]$

$= 3 + 9x + 27x^2 + 81x^3 + \ldots, \; |-3x| < 1 \;\; \therefore \; |x| < \frac{1}{3}$

$\therefore \; \text{f}(x) \equiv (1 - x + x^2 - x^3 + \ldots) + (3 + 9x + 27x^2 + 81x^3 + \ldots)$

$\text{f}(x) \equiv 4 + 8x + 28x^2 + 80x^3 + \ldots$, valid for $|x| < \frac{1}{3}$

b $\frac{1-6x}{1+3x-4x^2} \equiv \frac{1-6x}{(1-x)(1+4x)} \equiv \frac{A}{1-x} + \frac{B}{1+4x}$

$1 - 6x \equiv A(1+4x) + B(1-x)$

$x = 1 \quad \Rightarrow \quad -5 = 5A \quad \Rightarrow \quad A = -1$

$x = -\frac{1}{4} \quad \Rightarrow \quad \frac{5}{2} = \frac{5}{4}B \quad \Rightarrow \quad B = 2$

$\therefore \; \text{f}(x) \equiv \frac{2}{1+4x} - \frac{1}{1-x}$

$\frac{2}{1+4x} = 2(1+4x)^{-1} = 2[1 + (-1)(4x) + \frac{(-1)(-2)}{2}(4x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(4x)^3 + \ldots]$

$= 2 - 8x + 32x^2 - 128x^3 + \ldots, \; |4x| < 1 \;\; \therefore \; |x| < \frac{1}{4}$

$\frac{1}{1-x} = (1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(-x)^3 + \ldots$

$= 1 + x + x^2 + x^3 + \ldots, \; |-x| < 1 \;\; \therefore \; |x| < 1$

$\therefore \; \text{f}(x) \equiv (2 - 8x + 32x^2 - 128x^3 + \ldots) - (1 + x + x^2 + x^3 + \ldots)$

$\text{f}(x) \equiv 1 - 9x + 31x^2 - 129x^3 + \ldots$, valid for $|x| < \frac{1}{4}$

c $\frac{5}{2-3x-2x^2} \equiv \frac{5}{(1-2x)(2+x)} = \frac{A}{1-2x} + \frac{B}{2+x}$

$5 \equiv A(2+x) + B(1-2x)$

$x = \frac{1}{2} \quad \Rightarrow \quad 5 = \frac{5}{2}A \quad \Rightarrow \quad A = 2$

$x = -2 \quad \Rightarrow \quad 5 = 5B \quad \Rightarrow \quad B = 1$

$\therefore \; \text{f}(x) \equiv \frac{2}{1-2x} + \frac{1}{2+x}$

$\frac{2}{1-2x} = 2(1-2x)^{-1} = 2[1 + (-1)(-2x) + \frac{(-1)(-2)}{2}(-2x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(-2x)^3 + \ldots]$

$= 2 + 4x + 8x^2 + 16x^3 + \ldots, \; |-2x| < 1 \;\; \therefore \; |x| < \frac{1}{2}$

$\frac{1}{2+x} = (2+x)^{-1} = 2^{-1}(1 + \frac{1}{2}x)^{-1} = \frac{1}{2}[1 + (-1)(\frac{1}{2}x) + \frac{(-1)(-2)}{2}(\frac{1}{2}x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(\frac{1}{2}x)^3 + \ldots]$

$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 + \ldots, \; |\frac{1}{2}x| < 1 \;\; \therefore \; |x| < 2$

$\therefore \; \text{f}(x) \equiv (2 + 4x + 8x^2 + 16x^3 + \ldots) + (\frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 + \ldots)$

$\text{f}(x) \equiv \frac{5}{2} + \frac{15}{4}x + \frac{65}{8}x^2 + \frac{255}{16}x^3 + \ldots$, valid for $|x| < \frac{1}{2}$

Back to Previous Menu

d $\frac{7x-3}{x^2-4x+3} \equiv \frac{7x-3}{(x-1)(x-3)} \equiv \frac{A}{x-1} + \frac{B}{x-3}$

$7x - 3 \equiv A(x-3) + B(x-1)$

$x = 1 \Rightarrow 4 = -2A \Rightarrow A = -2$

$x = 3 \Rightarrow 18 = 2B \Rightarrow B = 9$

$\therefore \text{f}(x) \equiv \frac{9}{x-3} - \frac{2}{x-1} \equiv \frac{2}{1-x} - \frac{9}{3-x}$

$\frac{2}{1-x} = 2(1-x)^{-1} = 2[1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(-x)^3 + \ldots]$

$= 2 + 2x + 2x^2 + 2x^3 + \ldots, \ |-x| < 1 \ \therefore |x| < 1$

$\frac{9}{3-x} = 9(3-x)^{-1} = 9 \times 3^{-1}(1 - \frac{1}{3}x)^{-1}$

$= 3[1 + (-1)(-\frac{1}{3}x) + \frac{(-1)(-2)}{2}(-\frac{1}{3}x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(-\frac{1}{3}x)^3 + \ldots]$

$= 3 + x + \frac{1}{3}x^2 + \frac{1}{9}x^3 + \ldots, \ |-\frac{1}{3}x| < 1 \ \therefore |x| < 3$

$\therefore \text{f}(x) \equiv (2 + 2x + 2x^2 + 2x^3 + \ldots) - (3 + x + \frac{1}{3}x^2 + \frac{1}{9}x^3 + \ldots)$

$\text{f}(x) \equiv -1 + x + \frac{5}{3}x^2 + \frac{17}{9}x^3 + \ldots$, valid for $|x| < 1$

e $\frac{3+5x}{(1+3x)(1+x)^2} \equiv \frac{A}{1+3x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$

$3 + 5x \equiv A(1+x)^2 + B(1+3x)(1+x) + C(1+3x)$

$x = -\frac{1}{3} \Rightarrow \frac{4}{3} = \frac{4}{9}A \Rightarrow A = 3$

$x = -1 \Rightarrow -2 = -2C \Rightarrow C = 1$

coeffs of $x^2 \Rightarrow 0 = A + 3B \Rightarrow B = -1$

$\therefore \text{f}(x) \equiv \frac{3}{1+3x} - \frac{1}{1+x} + \frac{1}{(1+x)^2}$

$\frac{3}{1+3x} = 3(1+3x)^{-1} = 3[1 + (-1)(3x) + \frac{(-1)(-2)}{2}(3x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(3x)^3 + \ldots]$

$= 3 - 9x + 27x^2 - 81x^3 + \ldots, \ |3x| < 1 \ \therefore |x| < \frac{1}{3}$

$\frac{1}{1+x} = (1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3\times2}x^3 + \ldots$

$= 1 - x + x^2 - x^3 + \ldots, \ |x| < 1$

$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2}x^2 + \frac{(-2)(-3)(-4)}{3\times2}x^3 + \ldots$

$= 1 - 2x + 3x^2 - 4x^3 + \ldots, \ |x| < 1$

$\therefore \text{f}(x) \equiv (3 - 9x + 27x^2 - 81x^3 + \ldots) - (1 - x + x^2 - x^3 + \ldots) + (1 - 2x + 3x^2 - 4x^3 + \ldots)$

$\text{f}(x) \equiv 3 - 10x + 29x^2 - 84x^3 + \ldots$, valid for $|x| < \frac{1}{3}$

Back to Previous Menu

f

$$\begin{array}{r|l} & 1 \\ 2x^2 + x - 1 & 2x^2 + 0x + 4 \\ & 2x^2 + x - 1 \\ \hline & -x + 5 \end{array}$$

$\therefore \ \dfrac{2x^2+4}{2x^2+x-1} \equiv 1 + \dfrac{5-x}{2x^2+x-1}$

$\dfrac{5-x}{2x^2+x-1} \equiv \dfrac{5-x}{(2x-1)(x+1)} \equiv \dfrac{A}{2x-1} + \dfrac{B}{x+1}$

$5 - x \equiv A(x + 1) + B(2x - 1)$

$x = \frac{1}{2} \quad \Rightarrow \quad \frac{9}{2} = \frac{3}{2}A \quad \Rightarrow \quad A = 3$

$x = -1 \quad \Rightarrow \quad 6 = -3B \quad \Rightarrow \quad B = -2$

$\therefore \ \mathrm{f}(x) \equiv 1 + \dfrac{3}{2x-1} - \dfrac{2}{x+1} \equiv 1 - \dfrac{3}{1-2x} - \dfrac{2}{1+x}$

$\dfrac{3}{1-2x} = 3(1 - 2x)^{-1} = 3[1 + (-1)(-2x) + \frac{(-1)(-2)}{2}(-2x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-2x)^3 + \ldots]$

$= 3 + 6x + 12x^2 + 24x^3 + \ldots, \ |-2x| < 1 \ \therefore |x| < \frac{1}{2}$

$\dfrac{2}{1+x} = 2(1 + x)^{-1} = 2[1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3\times 2}x^3 + \ldots]$

$= 2 - 2x + 2x^2 - 2x^3 + \ldots, \ |x| < 1$

$\therefore \ \mathrm{f}(x) \equiv 1 - (3 + 6x + 12x^2 + 24x^3 + \ldots) - (2 - 2x + 2x^2 - 2x^3 + \ldots)$

$\mathrm{f}(x) \equiv -4 - 4x - 14x^2 - 22x^3 + \ldots$, valid for $|x| < \frac{1}{2}$

C4 SERIES — Answers - Worksheet B

1 **a** $= 1 + (\frac{1}{2})(-x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}(-x)^3 + \ldots$

$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \ldots$

b when $x = 0.01$, $(1 - x)^{\frac{1}{2}} \approx 1 - \frac{1}{2}(0.01) - \frac{1}{8}(0.01)^2 - \frac{1}{16}(0.01)^3$

$= 1 - 0.005 - 0.000\,012\,5 - 0.000\,000\,062\,5 = 0.994\,987\,437\,5$

$(1 - 0.01)^{\frac{1}{2}} = \sqrt{0.99} = \sqrt{\frac{9\times11}{100}} = \frac{3}{10}\sqrt{11}$

$\therefore \sqrt{11} = \frac{10}{3} \times 0.994\,987\,437\,5 = 3.316\,624\,79$ (9sf)

2 **a** $a = \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(8)^2 = -8$, $b = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}(8)^3 = 32$

b when $x = 0.01$, $(1 + 8x)^{\frac{1}{2}} \approx 1 + 4(0.01) - 8(0.01)^2 + 32(0.01)^3$

$= 1 + 0.04 - 0.000\,8 + 0.000\,032 = 1.039\,232$

$(1 + 0.08)^{\frac{1}{2}} = \sqrt{1.08} = \sqrt{\frac{36\times3}{100}} = \frac{3}{5}\sqrt{3}$

$\therefore \sqrt{3} = \frac{5}{3} \times 1.039\,232 = 1.732\,05$ (5dp)

3 **a** $= 9^{\frac{1}{2}}(1 - \frac{2}{3}x)^{\frac{1}{2}} = 3(1 - \frac{2}{3}x)^{\frac{1}{2}}$

$= 3[1 + (\frac{1}{2})(-\frac{2}{3}x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-\frac{2}{3}x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}(-\frac{2}{3}x)^3 + \ldots]$

$= 3 - x - \frac{1}{6}x^2 - \frac{1}{18}x^3 + \ldots$

b let $x = 0.05$

$\sqrt{8.7} \approx 3 - 0.05 - \frac{1}{6}(0.05)^2 - \frac{1}{18}(0.05)^3$

$= 2.949\,576$ (7sf)

4 **a** $= 1 + (\frac{1}{3})(6x) + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}(6x)^2 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3\times2}(6x)^3 + \ldots$

$= 1 + 2x - 4x^2 + \frac{40}{3}x^3 + \ldots$

b when $x = 0.004$, $(1 + 6x)^{\frac{1}{3}} \approx 1 + 2(0.004) - 4(0.004)^2 + \frac{40}{3}(0.004)^3$

$= 1.007\,936\,853$

$(1 + 0.024)^{\frac{1}{3}} = \sqrt[3]{1.024} = \sqrt[3]{\frac{512\times2}{1000}} = \frac{4}{5}\sqrt[3]{2}$

$\therefore \sqrt[3]{2} = \frac{5}{4} \times 1.007\,936\,853 = 1.259\,921$ (7sf)

5 **a** $= 1 + (-3)(2x) + \frac{(-3)(-4)}{2}(2x)^2 + \frac{(-3)(-4)(-5)}{3\times2}(2x)^3 + \ldots$

$= 1 - 6x + 24x^2 - 80x^3 + \ldots$, $|2x| < 1 \;\therefore$ valid for $|x| < \frac{1}{2}$

b $= (1 + 3x)(1 + 2x)^{-3} = (1 + 3x)(1 - 6x + 24x^2 - 80x^3 + \ldots)$

$= 1 - 6x + 24x^2 - 80x^3 + 3x - 18x^2 + 72x^3 + \ldots$

$= 1 - 3x + 6x^2 - 8x^3 + \ldots$

6 $\frac{2+x}{\sqrt{4-2x}} = (2 + x)(4 - 2x)^{-\frac{1}{2}} = (2 + x) \times 4^{-\frac{1}{2}}(1 - \frac{1}{2}x)^{-\frac{1}{2}}$

$= (2 + x) \times \frac{1}{2}[1 + (-\frac{1}{2})(-\frac{1}{2}x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-\frac{1}{2}x)^2 + \ldots]$

$= (2 + x)(\frac{1}{2} + \frac{1}{8}x + \frac{3}{64}x^2 + \ldots)$

$\therefore$ coeff of $x^2 = (2 \times \frac{3}{64}) + (1 \times \frac{1}{8}) = \frac{7}{32}$

Back to Previous Menu

7 **a** $\dfrac{2-11x}{1-5x+4x^2} \equiv \dfrac{A}{1-x} + \dfrac{B}{1-4x}$

$2 - 11x \equiv A(1 - 4x) + B(1 - x)$

$x = 1 \quad \Rightarrow \quad -9 = -3A \quad \Rightarrow \quad A = 3$

$x = \tfrac{1}{4} \quad \Rightarrow \quad -\tfrac{3}{4} = \tfrac{3}{4}B \quad \Rightarrow \quad B = -1$

b $\dfrac{2-11x}{1-5x+4x^2} \equiv \dfrac{3}{1-x} - \dfrac{1}{1-4x}$

$\dfrac{3}{1-x} = 3(1-x)^{-1} = 3[1 + (-1)(-x) + \tfrac{(-1)(-2)}{2}(-x)^2 + \tfrac{(-1)(-2)(-3)}{3\times 2}(-x)^3 + \ldots]$

$= 3 + 3x + 3x^2 + 3x^3 + \ldots, \ |-x| < 1 \ \therefore |x| < 1$

$\dfrac{1}{1-4x} = (1-4x)^{-1} = 1 + (-1)(-4x) + \tfrac{(-1)(-2)}{2}(-4x)^2 + \tfrac{(-1)(-2)(-3)}{3\times 2}(-4x)^3 + \ldots$

$= 1 + 4x + 16x^2 + 64x^3 + \ldots, \ |-4x| < 1 \ \therefore |x| < \tfrac{1}{4}$

$\therefore \ \dfrac{2-11x}{1-5x+4x^2} = (3 + 3x + 3x^2 + 3x^3 + \ldots) - (1 + 4x + 16x^2 + 64x^3 + \ldots)$

$= 2 - x - 13x^2 - 61x^3 + \ldots,$ valid for $|x| < \tfrac{1}{4}$

8 **a** $\dfrac{4-17x}{(1+2x)(1-3x)^2} \equiv \dfrac{A}{1+2x} + \dfrac{B}{1-3x} + \dfrac{C}{(1-3x)^2}$

$4 - 17x \equiv A(1-3x)^2 + B(1+2x)(1-3x) + C(1+2x)$

$x = -\tfrac{1}{2} \quad \Rightarrow \quad \tfrac{25}{2} = \tfrac{25}{4}A \quad \Rightarrow \quad A = 2$

$x = \tfrac{1}{3} \quad \Rightarrow \quad -\tfrac{5}{3} = \tfrac{5}{3}C \quad \Rightarrow \quad C = -1$

coeffs of $x^2 \quad \Rightarrow \quad 0 = 9A - 6B \quad \Rightarrow \quad B = 3$

$\therefore \ \mathrm{f}(x) \equiv \dfrac{2}{1+2x} + \dfrac{3}{1-3x} - \dfrac{1}{(1-3x)^2}$

b $\dfrac{2}{1+2x} = 2(1+2x)^{-1} = 2[1 + (-1)(2x) + \tfrac{(-1)(-2)}{2}(2x)^2 + \tfrac{(-1)(-2)(-3)}{3\times 2}(2x)^3 + \ldots]$

$= 2 - 4x + 8x^2 - 16x^3 + \ldots$

$\dfrac{3}{1-3x} = 3(1-3x)^{-1} = 3[1 + (-1)(-3x) + \tfrac{(-1)(-2)}{2}(-3x)^2 + \tfrac{(-1)(-2)(-3)}{3\times 2}(-3x)^3 + \ldots]$

$= 3 + 9x + 27x^2 + 81x^3 + \ldots$

$\dfrac{1}{(1-3x)^2} = (1-3x)^{-2} = 1 + (-2)(-3x) + \tfrac{(-2)(-3)}{2}(-3x)^2 + \tfrac{(-2)(-3)(-4)}{3\times 2}(-3x)^3 + \ldots$

$= 1 + 6x + 27x^2 + 108x^3 + \ldots$

$\mathrm{f}(x) = (2 - 4x + 8x^2 - 16x^3 + \ldots) + (3 + 9x + 27x^2 + 81x^3 + \ldots) - (1 + 6x + 27x^2 + 108x^3 + \ldots)$

$= 4 - x + 8x^2 - 43x^3 + \ldots$

9 **a** $(1 + ax)^b = 1 + b(ax) + \tfrac{b(b-1)}{2}(ax)^2 + \ldots$

$\therefore \quad ab = -6 \quad (1)$

and $\quad \tfrac{1}{2}a^2b(b-1) = 24 \quad (2)$

$(1) \quad \Rightarrow \quad a = -\dfrac{6}{b}$

sub. (2) $\quad \Rightarrow \quad \dfrac{18}{b}(b-1) = 24$

$18b - 18 = 24b$

$b = -3$

$a = 2$

b $= \tfrac{(-3)(-4)(-5)}{3\times 2}(2)^3 = -80$

Back to Previous Menu

1 **a** $= 1 + (\frac{1}{2})(-4x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-4x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times 2}(-4x)^3 + \ldots$

$= 1 - 2x - 2x^2 - 4x^3 + \ldots, \ |-4x| < 1 \quad \therefore$ valid for $|x| < \frac{1}{4}$

b when $x = 0.01$, $(1-4x)^{\frac{1}{2}} \approx 1 - 2(0.01) - 2(0.01)^2 - 4(0.01)^3$

$= 1 - 0.02 - 0.0002 - 0.000\,004$

$= 0.979\,796$

$(1 - 0.04)^{\frac{1}{2}} = \sqrt{0.96} = \sqrt{\frac{16\times 6}{100}} = \frac{2}{5}\sqrt{6}$

$\therefore \sqrt{6} \approx \frac{5}{2} \times 0.979\,796 = 2.44949$ (6sf)

2 **a** $\frac{4}{1+2x-3x^2} \equiv \frac{4}{(1+3x)(1-x)} \equiv \frac{A}{1+3x} + \frac{B}{1-x}$

$4 \equiv A(1-x) + B(1+3x)$

$x = -\frac{1}{3} \Rightarrow 4 = \frac{4}{3}A \Rightarrow A = 3$

$x = 1 \Rightarrow 4 = 4B \Rightarrow B = 1$

$\therefore \ \mathrm{f}(x) = \frac{3}{1+3x} + \frac{1}{1-x}$

b $\frac{3}{1+3x} = 3(1+3x)^{-1} = 3[1 + (-1)(3x) + \frac{(-1)(-2)}{2}(3x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(3x)^3 + \ldots]$

$= 3 - 9x + 27x^2 - 81x^3 + \ldots, \ |3x| < 1 \ \therefore$ valid for $|x| < \frac{1}{3}$

$\frac{1}{1-x} = (1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-x)^3 + \ldots$

$= 1 + x + x^2 + x^3 + \ldots, \ |-x| < 1 \ \therefore$ valid for $|x| < 1$

$\therefore \ \mathrm{f}(x) = (3 - 9x + 27x^2 - 81x^3 + \ldots) + (1 + x + x^2 + x^3 + \ldots)$

$= 4 - 8x + 28x^2 - 80x^3 + \ldots,$ valid for $|x| < \frac{1}{3}$

3 **a** $= 2^{-2}(1 - \frac{1}{2}x)^{-2} = \frac{1}{4}(1 - \frac{1}{2}x)^{-2}$

$= \frac{1}{4}[1 + (-2)(-\frac{1}{2}x) + \frac{(-2)(-3)}{2}(-\frac{1}{2}x)^2 + \frac{(-2)(-3)(-4)}{3\times 2}(-\frac{1}{2}x)^3 + \ldots]$

$= \frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 + \ldots$

b $\frac{3-x}{(2-x)^2} = (3-x)(2-x)^{-2} = (3-x)(\frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 + \ldots)$

$\therefore$ coefficient of $x^3 = (3 \times \frac{1}{8}) + (-1 \times \frac{3}{16}) = \frac{3}{16}$

4 **a** $\mathrm{f}(\frac{1}{10}) = \frac{4}{\sqrt{1+\frac{1}{15}}} = \frac{4}{\sqrt{\frac{16}{15}}} = \frac{4}{\frac{4}{\sqrt{15}}} = 4 \times \frac{\sqrt{15}}{4} = \sqrt{15}$

b $= 4(1 + \frac{2}{3}x)^{-\frac{1}{2}} = 4[1 + (-\frac{1}{2})(\frac{2}{3}x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\frac{2}{3}x)^2 + \ldots]$

$= 4 - \frac{4}{3}x + \frac{2}{3}x^2 + \ldots$

c $\sqrt{15} = \mathrm{f}(\frac{1}{10}) \approx 4 - \frac{4}{3}\times\frac{1}{10} + \frac{2}{3}\times(\frac{1}{10})^2 + \ldots$

$= 4 - \frac{2}{15} + \frac{1}{150} = 3\frac{131}{150}$

d $\sqrt{15} = 3.872\,98\ldots$

$3\frac{131}{150} = 3.873\,33\ldots$

$3\frac{55}{63} = 3.873\,01\ldots$

$\therefore \sqrt{15} < 3\frac{55}{63} < 3\frac{131}{150}$, so $3\frac{55}{63}$ is a more accurate approximation

Back to Previous Menu

5 **a** $= 1 + (\frac{1}{3})(-x) + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}(-x)^2 + \ldots$

$= 1 - \frac{1}{3}x - \frac{1}{9}x^2 + \ldots$

b when $x = 10^{-3}$, $(1 - x)^{\frac{1}{3}} \approx 1 - \frac{1}{3}(10^{-3}) - \frac{1}{9}(10^{-3})^2$

$= 0.999\,666\,555\,6$

$(1 - 10^{-3})^{\frac{1}{3}} = \sqrt[3]{0.999} = \sqrt[3]{\frac{27\times 37}{1000}} = \frac{3}{10}\sqrt[3]{37}$

$\therefore \sqrt[3]{37} \approx \frac{10}{3} \times 0.999\,666\,555\,6 = 3.332\,221\,85$ (9sf)

6 **a** $p = \frac{(\frac{3}{5})(-\frac{2}{5})}{2}(5)^2 = -3$

$q = \frac{(\frac{3}{5})(-\frac{2}{5})(-\frac{7}{5})}{3\times 2}(5)^3 = 7$

b let $x = 0.02$

$(1.1)^{\frac{3}{5}} \approx 1 + 3(0.02) - 3(0.02)^2 + 7(0.02)^3$

$= 1 + 0.06 - 0.0012 + 0.000\,056$

$= 1.058\,856$

c $(1.1)^{\frac{3}{5}} = 1.058\,852\,853\ldots$

% error $= \frac{1.058856 - 1.058852853}{1.058852853} \times 100\% = 0.000\,297$ % (3sf)

7 **a** $8 - 6x^2 \equiv A(2 + x)^2 + B(1 + x)(2 + x) + C(1 + x)$

$x = -1 \quad \Rightarrow \quad A = 2$

$x = -2 \quad \Rightarrow \quad -16 = -C \quad \Rightarrow \quad C = 16$

coeffs of $x^2 \quad \Rightarrow \quad -6 = A + B \quad \Rightarrow \quad B = -8$

b $\frac{8 - 6x^2}{(1+x)(2+x)^2} \equiv \frac{2}{1+x} - \frac{8}{2+x} + \frac{16}{(2+x)^2}$

$\frac{2}{1+x} = 2(1 + x)^{-1} = 2[1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3\times 2}x^3 + \ldots]$

$= 2 - 2x + 2x^2 - 2x^3 + \ldots$

$\frac{8}{2+x} = 8(2 + x)^{-1} = 8 \times 2^{-1}(1 + \frac{1}{2}x)^{-1} = 4(1 + \frac{1}{2}x)^{-1}$

$= 4[1 + (-1)(\frac{1}{2}x) + \frac{(-1)(-2)}{2}(\frac{1}{2}x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(\frac{1}{2}x)^3 + \ldots]$

$= 4 - 2x + x^2 - \frac{1}{2}x^3 + \ldots$

$\frac{16}{(2+x)^2} = 16(2 + x)^{-2} = 16 \times 2^{-2}(1 + \frac{1}{2}x)^{-2} = 4(1 + \frac{1}{2}x)^{-2}$

$= 4[1 + (-2)(\frac{1}{2}x) + \frac{(-2)(-3)}{2}(\frac{1}{2}x)^2 + \frac{(-2)(-3)(-4)}{3\times 2}(\frac{1}{2}x)^3 + \ldots]$

$= 4 - 4x + 3x^2 - 2x^3 + \ldots$

$\therefore \frac{8 - 6x^2}{(1+x)(2+x)^2} = (2 - 2x + 2x^2 - 2x^3 + \ldots) - (4 - 2x + x^2 - \frac{1}{2}x^3 + \ldots) + (4 - 4x + 3x^2 - 2x^3 + \ldots)$

$= 2 - 4x + 4x^2 - \frac{7}{2}x^3 + \ldots$

Back to Previous Menu

8 **a** $= 1 + (\tfrac{1}{2})(-2x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-2x)^2 + \ldots$

$= 1 - x - \tfrac{1}{2}x^2 + \ldots$

b when $x = 0.0008$, $(1 - 2x)^{\frac{1}{2}} \approx 1 - 0.0008 - \tfrac{1}{2}(0.0008)^2$

$= 1 - 0.0008 - 0.000\,000\,32$

$= 0.999\,199\,68$

$(1 - 0.0016)^{\frac{1}{2}} = \sqrt{0.9984} = \sqrt{\frac{256\times39}{10000}} = \frac{4}{25}\sqrt{39}$

$\therefore \sqrt{39} \approx \frac{25}{4} \times 0.999\,199\,68 = 6.244\,998$ (7sf)

9 **a** $= 1 + (\tfrac{1}{3})(8x) + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}(8x)^2 + \ldots$

$= 1 + \tfrac{8}{3}x - \tfrac{64}{9}x^2 + \ldots$

b $k = \sqrt[3]{\frac{5}{1.08}} = \sqrt[3]{\frac{500}{108}} = \sqrt[3]{\frac{125}{27}} = \frac{5}{3}$

c let $x = 0.01$, $\sqrt[3]{1.08} = 1 + \tfrac{8}{3}(0.01) - \tfrac{64}{9}(0.01)^2$

$= 1.025\,955\,556$

$\therefore \sqrt[3]{5} = \tfrac{5}{3} \times 1.025\,955\,556 = 1.710$ (4sf)

10 **a** $\text{f}(x) \equiv \frac{6x}{(x-1)(x-3)} \equiv \frac{A}{x-1} + \frac{B}{x-3}$

$6x \equiv A(x - 3) + B(x - 1)$

$x = 1 \Rightarrow 6 = -2A \Rightarrow A = -3$

$x = 3 \Rightarrow 18 = 2B \Rightarrow B = 9$

$\text{f}(x) \equiv \frac{9}{x-3} - \frac{3}{x-1}$

b $\text{f}(x) = \frac{3}{1-x} - \frac{9}{3-x}$

$\frac{3}{1-x} = 3(1 - x)^{-1} = 3[1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(-x)^3 + \ldots]$

$= 3 + 3x + 3x^2 + 3x^3 + \ldots$

$\frac{9}{3-x} = 9(3 - x)^{-1} = 9 \times 3^{-1}(1 - \tfrac{1}{3}x)^{-1} = 3(1 - \tfrac{1}{3}x)^{-1}$

$= 3[1 + (-1)(-\tfrac{1}{3}x) + \frac{(-1)(-2)}{2}(-\tfrac{1}{3}x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(-\tfrac{1}{3}x)^3 + \ldots]$

$= 3 + x + \tfrac{1}{3}x^2 + \tfrac{1}{9}x^3 + \ldots$

$\therefore \text{f}(x) = (3 + 3x + 3x^2 + 3x^3 + \ldots) - (3 + x + \tfrac{1}{3}x^2 + \tfrac{1}{9}x^3 + \ldots)$

$= 2x + \tfrac{8}{3}x^2 + \tfrac{26}{9}x^3 + \ldots$

$\therefore$ for small x, $\text{f}(x) \approx 2x + \tfrac{8}{3}x^2 + \tfrac{26}{9}x^3$

Back to Previous Menu

11 **a** $= 4^{\frac{1}{2}}(1 + \frac{1}{4}x)^{\frac{1}{2}} = 2(1 + \frac{1}{4}x)^{\frac{1}{2}} = 2[1 + (\frac{1}{2})(\frac{1}{4}x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(\frac{1}{4}x)^2 + \ldots]$

$= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \ldots, \quad |\frac{1}{4}x| < 1 \quad \therefore$ valid for $|x| < 4$

b when $x = \frac{1}{20}$, $(4 + x)^{\frac{1}{2}} \approx 2 + \frac{1}{4}(\frac{1}{20}) - \frac{1}{64}(\frac{1}{20})^2$

$= 2.012\,460\,938$

$(4 + \frac{1}{20})^{\frac{1}{2}} = \sqrt{\frac{81}{20}} = \sqrt{\frac{81\times5}{100}} = \frac{9}{10}\sqrt{5}$

$\therefore \sqrt{5} \approx \frac{10}{9} \times 2.012\,460\,938 = 2.236\,067\,71$ (9sf)

c $\sqrt{5} = 2.236\,067\,977\ldots$

$\therefore$ estimate is accurate to 7 significant figures

12 **a** $= 1 + (-\frac{1}{2})(2x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(2x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3\times2}(2x)^3 + \ldots$

$= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \ldots$

b $\frac{2-5x}{\sqrt{1+2x}} = (2 - 5x)(1 + 2x)^{-\frac{1}{2}} = (2 - 5x)(1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \ldots)$

$= 2 - 2x + 3x^2 - 5x^3 - 5x + 5x^2 - \frac{15}{2}x^3 + \ldots$

$= 2 - 7x + 8x^2 - \frac{25}{2}x^3 + \ldots$

$\therefore$ for small x, $\frac{2-5x}{\sqrt{1+2x}} = 2 - 7x + 8x^2 - \frac{25}{2}x^3$

c $2 - 5x = \sqrt{3} \times \sqrt{1+2x} = \sqrt{3+6x}$

$(2 - 5x)^2 = 3 + 6x$

$4 - 20x + 25x^2 = 3 + 6x$

$25x^2 - 26x + 1 = 0$

$(25x - 1)(x - 1) = 0$

$x = \frac{1}{25}, 1$

d let $x = \frac{1}{25}$

$\sqrt{3} \approx 2 - 7(\frac{1}{25}) + 8(\frac{1}{25})^2 - \frac{25}{2}(\frac{1}{25})^3$

$= 1.732$

13 **a** $= 1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3\times2}x^3 + \ldots$

$= 1 - x + x^2 - x^3 + \ldots$

b $= 1 - bx + b^2x^2 - b^3x^3 + \ldots$

c $\frac{1+ax}{1+bx} = (1 + ax)(1 + bx)^{-1} = (1 + ax)(1 - bx + b^2x^2 - b^3x^3 + \ldots)$

$= 1 - bx + b^2x^2 - b^3x^3 + ax - abx^2 + ab^2x^3 + \ldots$

$= 1 + (a - b)x + (b^2 - ab)x^2 + (ab^2 - b^3)x^3 + \ldots$

$\therefore \quad a - b = -4 \quad (1)$

and $\quad b^2 - ab = 12 \quad (2)$

$(1) \quad \Rightarrow \quad a = b - 4$

sub. (2) $\quad b^2 - b(b - 4) = 12$

$4b = 12$

$b = 3, \; a = -1$

d $= ab^2 - b^3 = -9 - 27 = -36$