IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper L

Difficulty Rating: 4.1475 /0.7557

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet “Mathematical Formulae and Statistical Tables” may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 19 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

$$f(x)=\frac{4x+1}{(1-2x)(1+x)},\ |x|<\frac{1}{2}.$$

a) Find the first four terms in the series expansion of $(1+x)^{-1}$. **(3)**

b) Hence, find the first four terms in the series expansion of $(1-2x)^{-1}$. **(2)**

c) Hence show that

$$f(x)\approx 1+5x+7x^2+17x^3,$$

stating the range of values of x for which the above approximation is valid. **(6)**

Question 2

The temperature of an oven is modelled by

$$\theta = 225-200\mathrm{e}^{-0.1t},\ t\geq 0$$

where θ °C is the temperature of the oven, t minutes after it was switched on.

a) State the highest temperature of the oven according to this model. **(1)**

b) Determine the value of t when the oven temperature reaches 125°C. **(3)**

c) Show clearly that

$$\frac{d\theta}{dt}=\frac{1}{10}(225-\theta),$$

and hence find the rate at which the temperature of the oven is increasing when its temperature has reached 125°C. **(4)**

Question 3

A circle has its centre at the point $C(-3,8)$ and the length of its diameter is $\sqrt{80}$.

a) State an equation for this circle. **(2)**

The straight line with equation

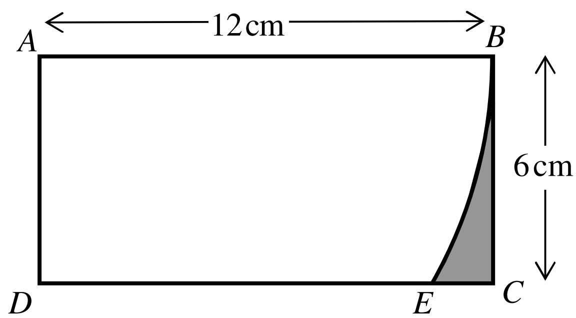
$$y = 3x + 7$$

intersects the circle at the points A and B.

b) Determine the coordinates of A and the coordinates of B. **(5)**

c) Show that ACB is a right angle and hence find the area of the triangle ACB. **(4)**

Question 4



The figure above shows the rectangle $ABCD$ where AB is 12 cm and BC is 6 cm.

An arc of a circle with centre at A and radius 12 cm is drawn inside the quadrilateral, meeting the side DC at the point E.

Find the area of the shaded region BEC. **(8)**

Question 5

A curve has equation

$$x^3 + xy + y^3 = 10 .$$

The straight line with equation $y = x + 2$ meets this curve at the point A.

a) Show that the x coordinate of A lies in the interval $(0.1, 0.2)$. **(5)**

b) Use the Newton Raphson method once, starting with $x = 0.1$, to find a better approximation for the x coordinate of A. **(5)**

Question 6

$$f(x) \equiv \ln x,\ x \in \mathbb{R},\ x > 0$$

$$g(x) \equiv 2\ln(x + \mathrm{e}),\ x \in \mathbb{R},\ x > -\mathrm{e} .$$

a) Describe mathematically the transformations which map the graph of $f(x)$ onto the graph of $g(x)$. **(3)**

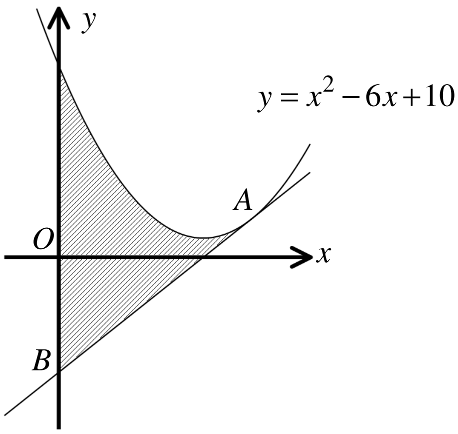
b) Sketch the graph of $y = |g(x)|$, indicating the coordinates of any intercepts of the graph with the coordinate axes. **(3)**

c) Solve the equation

$$|g(x)| = 2 . \quad \textbf{(4)}$$

d) Hence solve the inequality $|g(x)| \geq 2$. **(2)**

Question 7



The figure above shows the curve with equation

$$y = x^2 - 6x + 10, \ x \in \mathbb{R}.$$

The point A, where $x = 4$, lies on the curve.

The tangent to the curve at A, meets the y axis at B.

Determine the area of the finite region bounded by the curve, the tangent to the curve at A and the y axis. **(9)**

Question 8

A curve is defined implicitly by

$$4xy - (x+2)^2 = y^2 - 5.$$

a) Find a simplified expression for $\frac{dy}{dx}$, in terms of x and y. **(4)**

b) Hence determine the coordinates of the two stationary points of the curve. **(6)**

Question 9

Show with a detailed method that

$$\frac{\sqrt[3]{16}-\sqrt[3]{2}}{\sqrt[3]{4}} = k\sqrt[3]{4}$$

where k is a constant to be found. **(6)**

Question 10

At time t seconds, a spherical balloon has radius r cm and surface area S cm^2.

The surface area of the balloon is increasing at a constant rate of 24π cm^{2}s^{-1}.

a) Show that

$$\frac{dr}{dt}=\frac{3}{r}.$$ **(3)**

At time t seconds the balloon has volume V cm^3.

b) By considering $\frac{dV}{dr}\times\frac{dr}{dt}$, show further that

$$\frac{dV}{dt}=\sqrt[3]{1296\pi^2 V}.$$ **(4)**

c) Solve the differential equation of part **(b)** to show

$$V^{\frac{2}{3}}=\frac{2}{3}\left(1296\pi^2\right)^{\frac{1}{3}}t + \text{constant}.$$ **(4)**

d) Given that the initial volume of the balloon was 64π cm^3, find an exact simplified value of V when $t=\sqrt[3]{36}$. **(4)**

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3\right]$

$\left[\text{surface area of a sphere of radius } r \text{ is given by } 4\pi r^2\right]$

Question 11

Use the substitution $u = 1 + x^2 \operatorname{cosec} x$ to find an expression for

$$\int \frac{2x - x^2 \cot x}{x^2 + \sin x} \, dx. \qquad (7)$$

Question 12

$$f(x) \equiv Ax^5 + Bx^4 + 8x^2,$$

where A and B are non zero constants.

The polynomial $f(x)$ satisfies the relationship

$$f(x) \equiv (2x-1)(x-2)g(x) + 169x - 82.$$

a) Find the value of A and the value of B. **(5)**

b) Determine the polynomial $g(x)$. **(4)**

The polynomial $f(x)$ also satisfies the relationship

$$f(x) \equiv (x+2)^2 h(x) + Px + Q,$$

where P and Q are constants.

c) Find the value of each of the constants P and Q. **(4)**

Question 13

A curve C has equation

$$y = \frac{1}{x-1}, \quad x \neq 1.$$

a) Sketch the graph of C, clearly labelling its asymptote and the coordinates of any point where C meets the coordinate axes. **(3)**

The line with equation $y = a - 2x$, where a is a constant, does not meet C.

b) Show clearly that

$$2 - 2\sqrt{2} < a < 2 + 2\sqrt{2}.$$ **(7)**

Question 14

A curve C_1 has equation

$$y = \ln\sqrt{x} + \sqrt{\ln x}, \quad x > 1.$$

a) Differentiate y with respect to x, simplifying the answer as far as possible. **(3)**

A different curve C_2 has equation

$$y = (2x+1)^{\frac{1}{2}}(1-4x)^{-\frac{1}{2}}, \quad -\frac{1}{2} \leq x < \frac{1}{4}.$$

b) Show that C_2 has no turning points. **(6)**

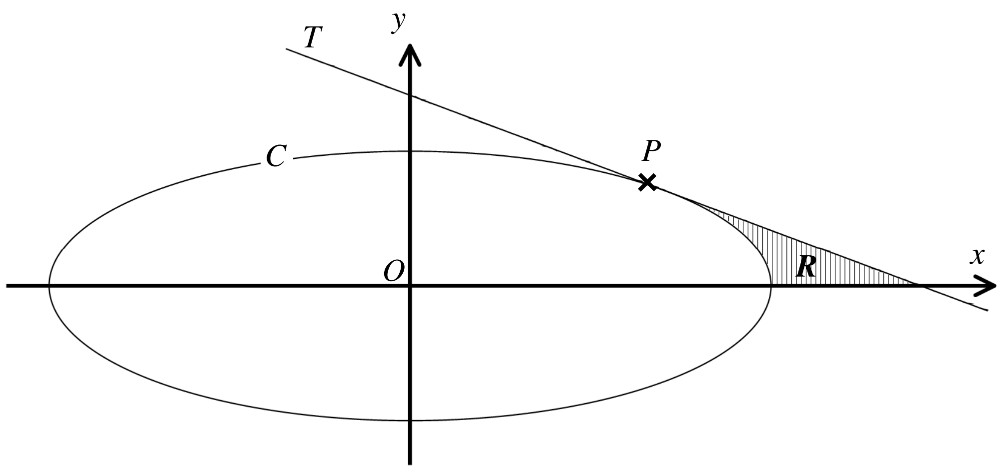
A third curve C_3 has equation

$$y = \frac{2x-1}{\sqrt{2x+1}}, \quad x \geq -\frac{1}{2}.$$

c) Show that

$$\frac{d}{dx}\left(\frac{2x-1}{\sqrt{2x+1}}\right) = \frac{2x+3}{(2x+1)^{\frac{3}{2}}}.$$ **(5)**

Question 15



The figure above shows the ellipse with parametric equations

$$x = 8\cos\theta,\ y = 4\sin\theta,\ 0 \le \theta < 2\pi .$$

The point P lies on the ellipse, where $\theta = \frac{1}{4}\pi$.

The straight line T is a tangent to the ellipse at P.

The finite region R, shown shaded in the figure, is bounded by the ellipse, the tangent T and the x axis.

Find an exact value for the area of R. **(12)**

Question 16

$$f(x) = \sqrt{1+x^2},\ x \in \mathbb{R} .$$

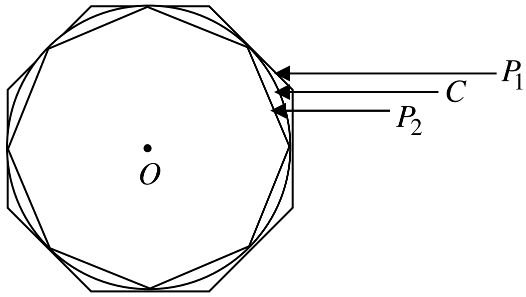
Use the formal definition of the derivative as a limit, to show that

$$f'(x) = \frac{x}{\sqrt{1+x^2}} .$$ **(7)**

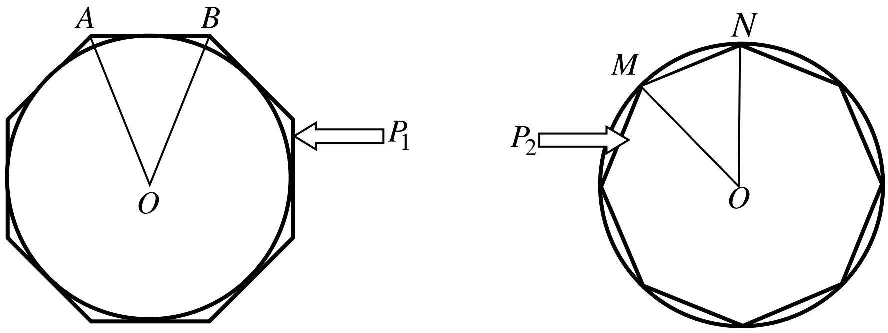
Question 17

The figure below shows a regular octagon P_1. A circle C is inscribed inside P_1 and another regular octagon P_2 is inscribed inside the circle C.

The three objects have a common centre at O.



The circle C has a radius of 1 unit. The points A and B are consecutive vertices of P_1, and the points M and N are consecutive vertices of P_2.



a) By considering the triangle OAB, show that the perimeter of the octagon P_1 is $16\tan\dfrac{\pi}{8}$. **(3)**

b) Use the triangle OMN in a similar fashion to show that the perimeter of the octagon P_2 is $16\sin\dfrac{\pi}{8}$. **(3)**

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c) Use a standard identity for $\cos 2\theta$ to show that

$$\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}\,. \qquad (5)$$

d) Show further that

$$\tan\frac{\pi}{8} = -1+\sqrt{2}\,. \qquad (5)$$

e) Deduce from the results obtained so far that

$$3.06 < \pi < 3.31\,. \qquad (2)$$

Question 18

The piecewise continuous function f is defined by

$$f(x) \equiv \begin{cases} \frac{5}{2} - x, & x \in \mathbb{R},\ -10 < x < 2 \\ \frac{2}{x^2}, & x \in \mathbb{R},\quad 2 \le x \le 4 \end{cases}$$

Determine an expression, similar to the one above, for the inverse of f.

You must also give the range of the inverse of f. **(10)**

Question 19

Consider the following 2 sequences.

$$10, 13, 16, 19, 22, \ldots \qquad \text{and} \qquad 6, 12, 24, 48, 96, \ldots$$

The sum of the n^{th} term of the first sequence and the n^{th} term of the second sequence is denoted by U_n.

Show algebraically that

$$U_{n+1} = U_n + 3\left(1+2^n\right). \qquad (9)$$