FP2 Functions

1. June 2010 qu. 6

(i) Show that $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$. [2]

(ii) Given that $y = \cosh(a \sinh^{-1} x)$, where a is a constant, show that

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - a^2y = 0.$$ [5]

2. June 2010 qu. 8

(i) Using the definition of $\cosh x$ in terms of e^x and e^{-x}, show that

$$4\cosh^3 x - 3\cosh x \equiv \cosh 3x.$$ [4]

(ii) Use the substitution $u = \cosh x$ to find, in terms of $5^{\frac{1}{3}}$, the real root of the equation

$$20u^3 - 15u - 13 = 0.$$ [6]

3. Jan 2010 qu.5

(i) Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x}, show that

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

Deduce that $1 - \tanh^2 x \equiv \operatorname{sech}^2 x$. [4]

(ii) Solve the equation $2\tanh^2 x - \operatorname{sech} x = 1$, giving your answer(s) in logarithmic form. [4]

4. Jan 2010 qu.9

(i) Given that $y = \tanh^{-1} x$, for $-1 < x < 1$, prove that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. [3]

(ii) It is given that $f(x) = a\cosh x - b\sinh x$, where a and b are positive constants.

(a) Given that $b \geq a$, show that the curve with equation $y = f(x)$ has no stationary points.

(b) In the case where $a > 1$ and $b = 1$, show that $f(x)$ has a minimum value of $\sqrt{a^2-1}$. [6]

5. June 2009 qu. 6

Given that $\int_0^1 \frac{1}{\sqrt{16+9x^2}}\,dx + \int_0^2 \frac{1}{\sqrt{9+4x^2}}\,dx = \ln a$, find the exact value of a. [6]

6. June 2009 qu. 7

(i) Sketch the graph of $y = \coth x$, and give the equations of any asymptotes. [3]

(ii) It is given that $f(x) = x \tanh x - 2$. Use the Newton-Raphson method, with a first approximation $x_1 = 2$, to find the next three approximations x_2, x_3 and x_4 to a root of $f(x) = 0$. Give the answers correct to 4 decimal places. [4]

(iii) If $f(x) = 0$, show that $\coth x = \frac{1}{2}x$. Hence write down the roots of $f(x) = 0$, correct to 4 decimal places. [3]

7. June 2009 qu. 8

(i) Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x}, show that

(a) $\cosh(\ln a) \equiv \dfrac{a^2+1}{2a}$, where $a > 0$, [3]

(b) $\cosh x \cosh y - \sinh x \sinh y \equiv \cosh(x - y)$. [3]

(ii) Use part (i)(b) to show that $\cosh^2 x - \sinh^2 x \equiv 1$. [1]

iii) Given that $R > 0$ and $a > 1$, find R and a such that

$13 \cosh x - 5 \sinh x \equiv R \cosh(x - \ln a)$. [5]

(iv) Hence write down the coordinates of the minimum point on the curve with equation $y = 13 \cosh x - 5 \sinh x$. [2]

8. Jan 2009 qu.6

(i) Using the definitions of $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}, show that

$$1 + 2 \sinh^2 x \equiv \cosh 2x.$$ [3]

(ii) Solve the equation $\cosh 2x - 5 \sinh x = 4$, giving your answers in logarithmic form. [5]

9. June 2008 qu. 4

(i) Sketch, on the same diagram, the curves with equations $y = \operatorname{sech} x$ and $y = x^2$. [3]

(ii) By using the definition of $\operatorname{sech} x$ in terms of e^x and e^{-x}, show that the x-coordinates of the points at which these curves meet are solutions of the equation $x^2 = \dfrac{2e^x}{e^{2x}+1}$. [3]

(iii) The iteration $x_{n+1} = \sqrt{\dfrac{2e^{x_n}}{e^{2x_n}+1}}$

can be used to find the positive root of the equation in part (ii). With initial value $x_1 = 1$, the approximations $x_2 = 0.8050$, $x_3 = 0.8633$, $x_4 = 0.8463$ and $x_5 = 0.8513$ are obtained, correct to 4 decimal places. State with a reason whether, in this case, the iteration produces a 'staircase' or a 'cobweb' diagram. [2]

10. June 2008 qu. 7

It is given that $f(x) = \tanh^{-1}\left(\frac{1-x}{2+x}\right)$, for $x > -\frac{1}{2}$.

(i) Show that $f'(x) = \frac{1}{1+2x}$, and find $f''(x)$. [6]

(ii) Show that the first three terms of the Maclaurin series for $f(x)$ can be written as $\ln a + bx + cx^2$, for constants a, b and c to be found. [4]

11. June 2008 qu. 8

(i) By using the definition of $\sinh x$ in terms of e^x and e^{-x}, show that

$$\sinh^3 x = \frac{1}{4}\sinh 3x - \frac{3}{4}\sinh x.$$ [4]

(ii) Find the range of values of the constant k for which the equation

$\sinh 3x = k \sinh x$ has real solutions other than $x = 0$. [3]

(iii) Given that $k = 4$, solve the equation in part (ii), giving the non-zero answers in logarithmic form. [3]

12. June 2007 qu. 7

(i) Using the definitions of hyperbolic functions in terms of exponentials, prove that

$$\cosh x \cosh y - \sinh x \sinh y = \cosh(x - y).$$ [4]

(ii) Given that $\cosh x \cosh y = 9$ and $\sinh x \sinh y = 8$, show that $x = y$. [2]

(iii) Hence find the values of x and y which satisfy the equations given in part (ii), giving the answers in logarithmic form. [4]

13. Jan 2007 qu.4

(i) On separate diagrams, sketch the graphs of $y = \sinh x$ and $y = \operatorname{cosech} x$. [3]

(ii) Show that $\operatorname{cosech} x = \frac{2e^x}{e^{2x} - 1}$, and hence, using the substitution $u = e^x$, find $\int \operatorname{cosech} x \, dx$. [6]

14. Jan 2007 qu.8

(i) Define tanh y in terms of e^y and e^{-y}. [1]

(ii) Given that $y = \tanh^{-1}x$, where $-1 < x < 1$, prove that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. [3]

(iii) Find the exact solution of the equation $3\cosh x = 4\sinh x$, giving the answer in terms of a logarithm. [2]

(iv) Solve the equation $\tanh^{-1}x + \ln(1 - x) = \ln\left(\frac{4}{5}\right)$. [3]

15. June 2006 qu. 4

(i) Using the definition of cosh x in terms of e^x and e^{-x}, prove that $\cosh 2x = 2\cosh^2 x - 1$ [3]

(ii) Hence solve the equation $\cosh 2x - 7\cosh x = 3$, giving your answer in logarithmic form. [4]

16. Jan 2006 qu.9

(i) Using the definitions of cosh$x$ and sinh$x$ in terms of e^x and e^{-x}, prove that

$$\sinh 2x = 2\sinh x \cosh x.$$ [4]

(ii) Show that the curve with equation $y = \cosh 2x - 6\sinh x$

has just one stationary point, and find its x-coordinate in logarithmic form.

Determine the nature of the stationary point. [8]