1.

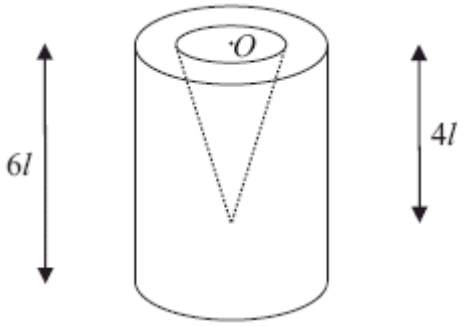


Figure 1

A container is formed by removing a right circular solid cone of height $4l$ from a uniform solid right circular cylinder of height $6l$. The centre O of the plane face of the cone coincides with the centre of a plane face of the cylinder and the axis of the cone coincides with the axis of the cylinder, as shown in Figure 1. The cylinder has radius $2l$ and the base of the cone has radius l.

(a) Find the distance of the centre of mass of the container from O.

(6)

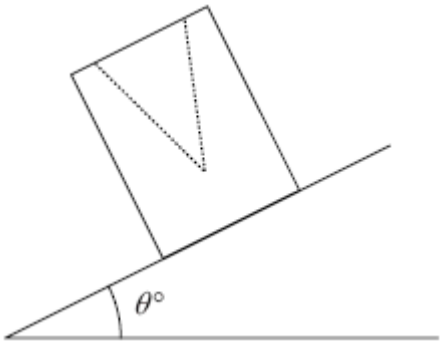


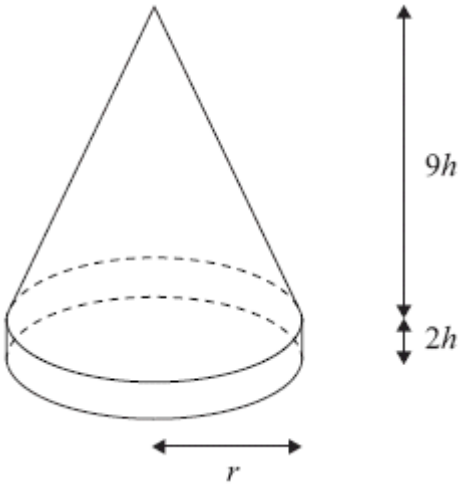
Figure 2

The container is placed on a plane which is inclined at an angle $\theta°$ to the horizontal. The open face is uppermost, as shown in Figure 2. The plane is sufficiently rough to prevent the container from sliding. The container is on the point of toppling.

(b) Find the value of θ.

(4)
(Total 10 marks)

2. [*The centre of mass of a uniform hollow cone of height h is* $\frac{1}{3}h$ *above the base on the line from the centre of the base to the vertex.*]



A marker for the route of a charity walk consists of a uniform hollow cone fixed on to a uniform solid cylindrical ring, as shown in the diagram above. The hollow cone has base radius r, height $9h$ and mass m. The solid cylindrical ring has outer radius r, height $2h$ and mass $3m$. The marker stands with its base on a horizontal surface.

(a) Find, in terms of h, the distance of the centre of mass of the marker from the horizontal surface.

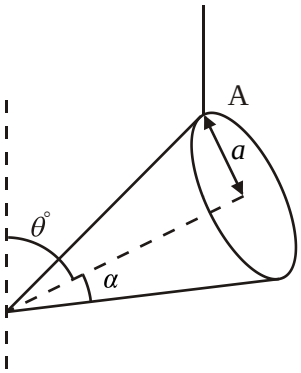
(5)

When the marker stands on a plane inclined at arctan $\frac{1}{12}$ to the horizontal it is on the point of toppling over. The coefficient of friction between the marker and the plane is large enough to be certain that the marker will not slip.

(b) Find h in terms of r.

(3)
(Total 8 marks)

3.

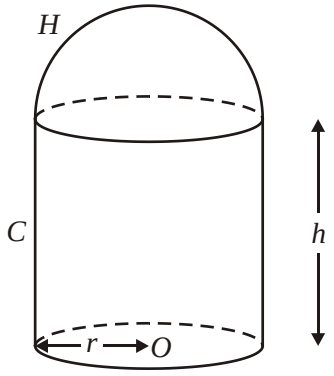


A uniform solid right circular cone has base radius a and semi-vertical angle α, where $\tan \alpha = \frac{1}{3}$. The cone is freely suspended by a string attached at a point A on the rim of its base, and hangs in equilibrium with its axis of symmetry making an angle of θ° with the upward vertical, as shown in the diagram above.

Find, to one decimal place, the value of θ.

(Total 5 marks)

4.



A body consists of a uniform solid circular cylinder C, together with a uniform solid hemisphere H which is attached to C. The plane face of H coincides with the upper plane face of C, as shown in the figure above. The cylinder C has base radius r, height h and mass $3M$. The mass of H is $2M$. The point O is the centre of the base of C.

(a) Show that the distance of the centre of mass of the body from O is

$$\frac{14h + 3r}{20}.$$

(5)

The body is placed with its plane face on a rough plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{4}{3}$. The plane is sufficiently rough to prevent slipping. Given that the body is on the point of toppling,

(b) find h in terms of r.

(4)
(Total 9 marks)

5. A closed container C consists of a thin uniform hollow hemispherical bowl of radius a, together with a lid. The lid is a thin uniform circular disc, also of radius a. The centre O of the disc coincides with the centre of the hemispherical bowl. The bowl and its lid are made of the same material.

(a) Show that the centre of mass of C is at a distance $\frac{1}{3}a$ from O.

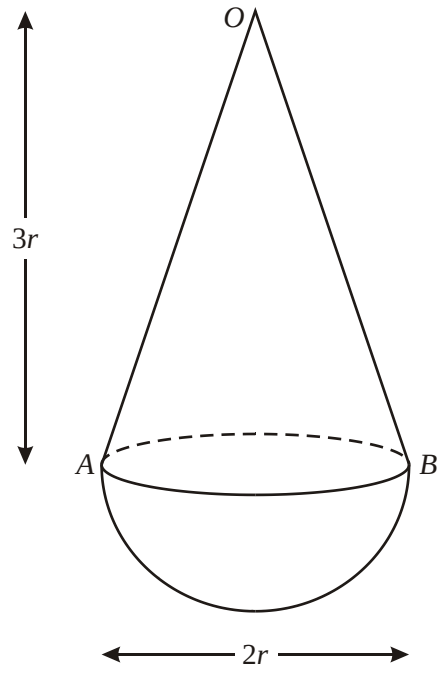
(4)

The container C has mass M. A particle of mass $\frac{1}{2}M$ is attached to the container at a point P on the circumference of the lid. The container is then placed with a point of its curved surface in contact with a horizontal plane. The container rests in equilibrium with P, O and the point of contact in the same vertical plane.

(b) Find, to the nearest degree, the angle made by the line PO with the horizontal.

(5)
(Total 9 marks)

6.



A child's toy consists of a uniform solid hemisphere, of mass M and base radius r, joined to a uniform solid right circular cone of mass m, where $2m < M$. The cone has vertex O, base radius r and height $3r$. Its plane face, with diameter AB, coincides with the plane face of the hemisphere, as shown in the diagram above.

(a) Show that the distance of the centre of mass of the toy from AB is

$$\frac{3(M-2m)}{8(M+m)}r.$$

(5)

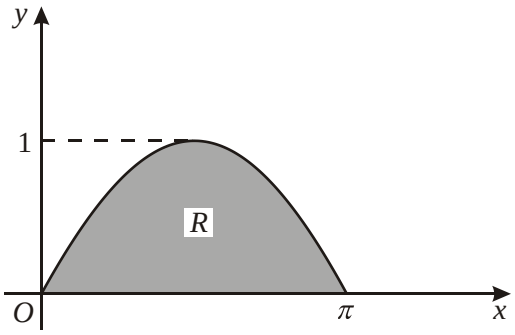
The toy is placed with OA on a horizontal surface. The toy is released from rest and does not remain in equilibrium.

(b) Show that $M > 26m$.

(4)
(Total 9 marks)

7.

Figure 1



A uniform lamina occupies the region R bounded by the x-axis and the curve

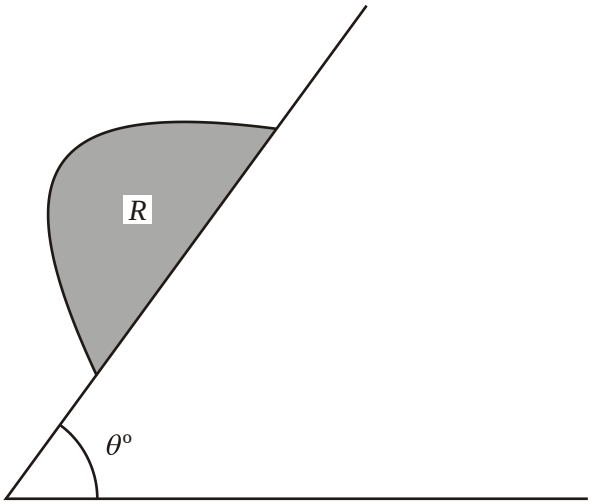
$$y = \sin x, \qquad 0 \leq x \leq \pi,$$

as shown in Figure 1.

(a) Show, by integration, that the y-coordinate of the centre of mass of the lamina is $\frac{\pi}{8}$.

(6)

Figure 2

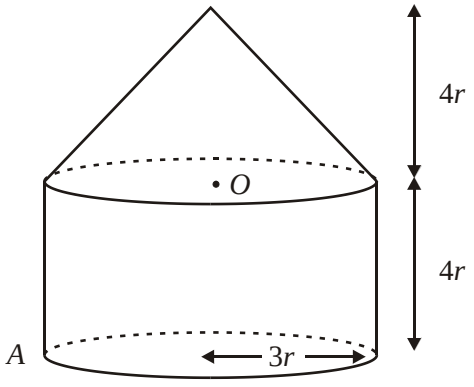


A uniform prism S has cross-section R. The prism is placed with its rectangular face on a table which is inclined at an angle θ to the horizontal. The cross-section R lies in a vertical plane as shown in Figure 2. The table is sufficiently rough to prevent S sliding. Given that S does not topple,

(b) find the largest possible value of θ.

(3)
(Total 9 marks)

8.



A toy is formed by joining a uniform solid right circular cone, of base radius $3r$ and height $4r$, to a uniform solid cylinder, also of radius $3r$ and height $4r$. The cone and the cylinder are made from the same material, and the plane face of the cone coincides with a plane face of the cylinder, as shown in the diagram above. The centre of this plane face is O.

(a) Find the distance of the centre of mass of the toy from O.

(5)

The point A lies on the edge of the plane face of the cylinder which forms the base of the toy. The toy is suspended from A and hangs in equilibrium.

(b) Find, in degrees to one decimal place, the angle between the axis of symmetry of the toy and the vertical.

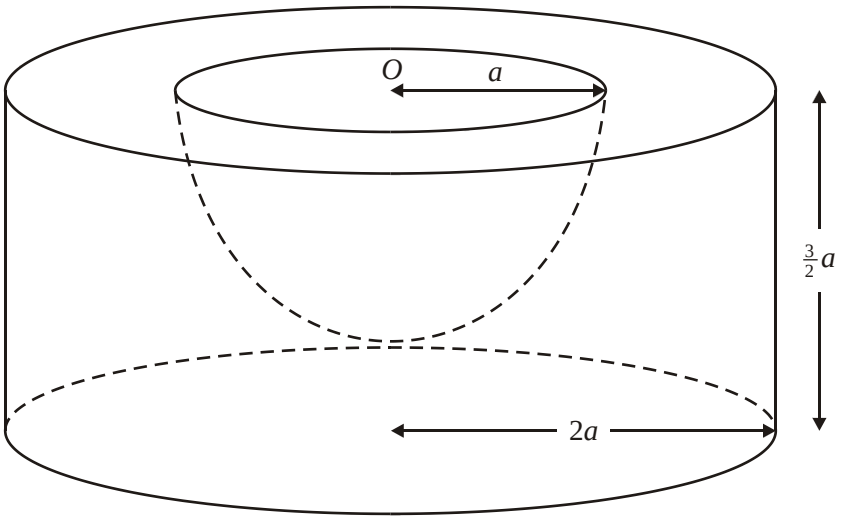
(4)

The toy is placed with the curved surface of the cone on horizontal ground.

(c) Determine whether the toy will topple.

(4)
(Total 13 marks)

9.



A uniform solid cylinder has radius $2a$ and height $\frac{3}{2}a$. A hemisphere of radius a is removed from the cylinder. The plane face of the hemisphere coincides with the upper plane face of the cylinder, and the centre O of the hemisphere is also the centre of this plane face, as shown in the diagram above. The remaining solid is S.

(a) Find the distance of the centre of mass of S from O.

(6)

The lower plane face of S rests in equilibrium on a desk lid which is inclined at an angle θ to the horizontal. Assuming that the lid is sufficiently rough to prevent S from slipping, and that S is on the point of toppling when $\theta = \alpha$,

(b) find the value of α.

(3)

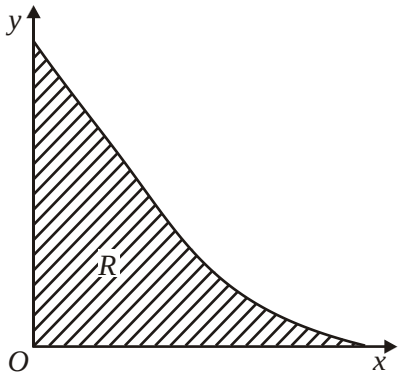
Given instead that the coefficient of friction between S and the lid is 0.8, and that S is on the point of sliding down the lid when $\theta = \beta$,

(c) find the value of β.

(3)
(Total 12 marks)

10.

Figure 1



The shaded region R is bounded by part of the curve with equation $y = \frac{1}{2}(x-2)^2$, the x-axis and the y-axis, as shown in Fig. 1. The unit of length on both axes is 1 cm. A uniform solid S is made by rotating R through 360° about the x-axis. Using integration,

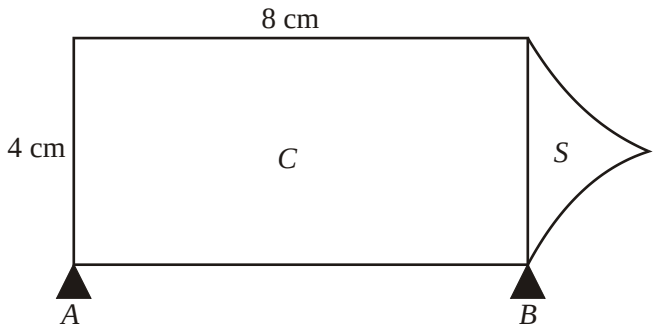
(a) calculate the volume of the solid S, leaving your answer in terms of π,

(4)

(b) show that the centre of mass of S is $\frac{1}{3}$ cm from its plane face.

(7)

Figure 2

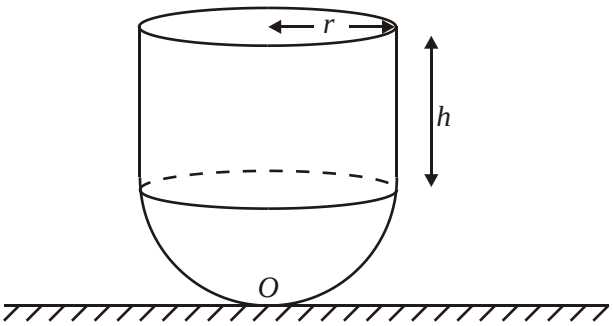


A tool is modelled as having two components, a solid uniform cylinder *C* and the solid *S*. The diameter of *C* is 4 cm and the length of *C* is 8 cm. One end of *C* coincides with the plane face of *S*. The components are made of different materials. The weight of *C* is 10*W* newtons and the weight of *S* is 2*W* newtons. The tool lies in equilibrium with its axis of symmetry horizontal on two smooth supports *A* and *B*, which are at the ends of the cylinder, as shown in Fig. 2.

(c) Find the magnitude of the force of the support *A* on the tool.

(5)
(Total 16 marks)

11.



A child's toy consists of a uniform solid hemisphere attached to a uniform solid cylinder. The plane face of the hemisphere coincides with the plane face of the cylinder, as shown in the diagram above. The cylinder and the hemisphere each have radius *r*, and the height of the cylinder is *h*. The material of the hemisphere is 6 times as dense as the material of the cylinder. The toy rests in equilibrium on a horizontal plane with the cylinder above the hemisphere and the axis of the cylinder vertical.

(a) Show that the distance *d* of the centre of mass of the toy from its lowest point *O* is given by

$$d = \frac{h^2 + 2hr + 5r^2}{2(h + 4r)}.$$

(7)

When the toy is placed with any point of the curved surface of the hemisphere resting on the plane it will remain in equilibrium.

(b) Find h in terms of r.

(3)
(Total 10 marks)

1

(a)

	cone	container	cylinder
mass ratio	$\frac{4\pi l^3}{3}$	$\frac{68\pi l^3}{3}$	$24\pi l^3$
	4	68	72
dist from *O*	l	$\bar{x}$	$3l$

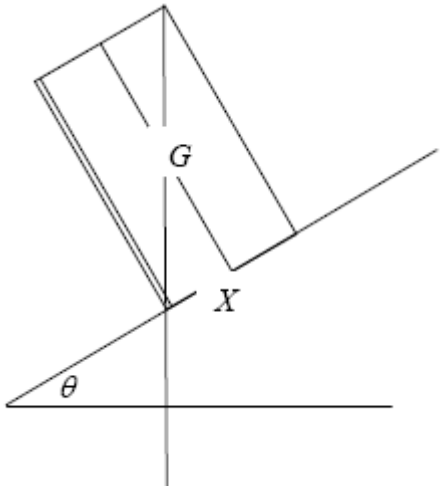
M1 A1

B1

Moments: $4l + 68\bar{x} = 72 \times 3l$ M1 A1ft

$\bar{x} = \frac{212l}{68} = \frac{53}{17}l$ accept $3.12l$ A1 6

(b)



$GX = 6l - \bar{x}$ seen M1

$\tan\theta = \frac{2l}{6l - \bar{x}}$ M1 A1

$= \frac{2 \times 17}{49}$

$\theta = 34.75... = 34.8$ or 35 A1 4

[10]

2. (a)

Object	Mass	c of m above base
Cone	m	$2h+3h$
Base	$3m$	h
Marker	$4m$	d

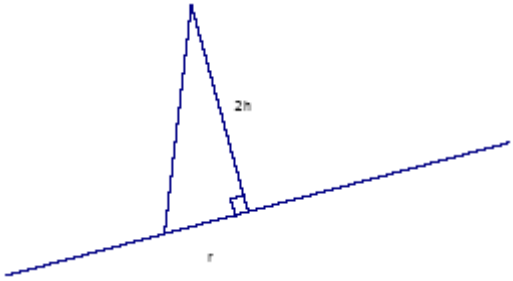
B1(ratio masses)

B1(distances)

$m \times 5h + 3m \times h = 4m \times d$ M1A1ft

$d = 2h$ A1

(b)

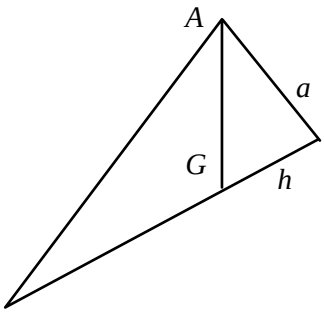


$\frac{r}{d} = \frac{1}{12}$ M1A1ft

$6r = h$ A1

[8]

3.



Height of cone = $\frac{a}{\tan\alpha} = 3a$ M1A1

Hence $h = \frac{3}{4}a$ M1

$\tan\theta = \frac{a}{\frac{3}{4}a} = \frac{4}{3} \Rightarrow \theta = 53.1°$ M1A1 5

1st M1 (generous) allow any trig ratio to get height of cone (e.g. using sin)

3rd M1 For correct trig ratio on a suitable triangle to get θ or complement (even if they call the angle by another name – hence if they are aware or not that they are getting the required angle)

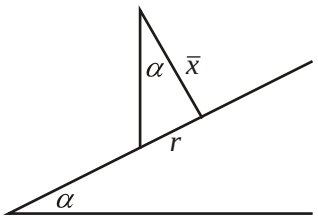
[5]

4. (a) $5M\bar{x} = 3M \times \frac{h}{2} + 2M\left(h + \frac{3}{8}r\right)$ M1 A2(1, 0)

$5\bar{x} = \frac{3h}{2} + 2h + \frac{3}{4}r = \frac{7h}{2} + \frac{3}{4}r$

$\bar{x} = \frac{14h + 3r}{20}$ cso M1 A1 5

(b)



$\tan\alpha = \frac{20r}{14h + 3r} = \frac{4}{3}$ M1 A1

Leading to $h = \frac{6}{7}r$ M1 A1 4

[9]

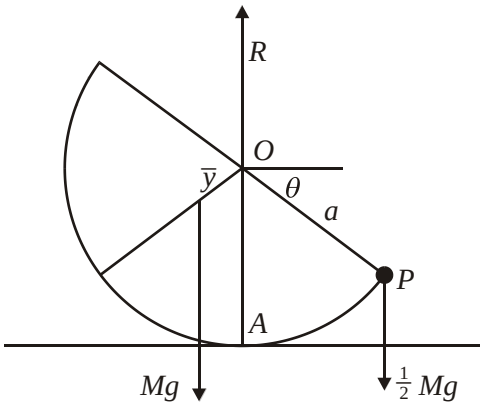
5. (a)

	Bowl	Lid	*C*		
Mass ratio	2	1	3	anything in ratio 2 : 1 : 3	B1
$\bar{y}$	$\frac{1}{2}a$	0	$\bar{y}$		B1

M(*O*) $2 \times \frac{1}{2}a = 3\bar{y}$ M1

$\bar{y} = \frac{1}{3}a$ (*) csoA1 4

(b)



M(*A*) $Mg \times \frac{1}{3}a\sin\theta = \frac{1}{2}Mg \times a\cos\theta$ M1 A1=A1

$\tan\theta = \frac{3}{2}$ M1

$\theta \approx 56°$ caoA1 5

[9]

Methods involving the location of the combined centre of mass of C and P.

G is the centre of mass of *C*; *G′* is the combined centre of mass of *C* and *P*.

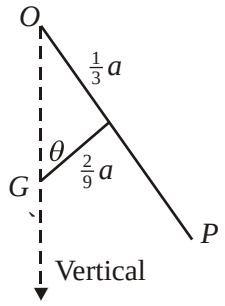
First Alternative

	C	*P*	*C* and *P*
Mass ratios	2	1	3
$\bar{y}$	$\frac{1}{3}a$	0	$\bar{y}$
$\bar{x}$	0	*a*	$\bar{x}$

Finding both coordinates of *G′* M1

$\frac{2}{3}a = 3\bar{y} \Rightarrow \bar{y} = \frac{2}{9}a$ A1

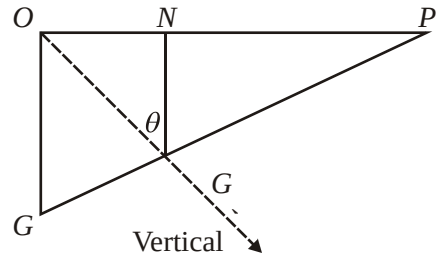
$a = 3\bar{x} \Rightarrow \bar{x} = \frac{1}{3}a$ A1



$\tan\theta = \frac{\frac{1}{3}a}{\frac{2}{9}a} = \frac{3}{2}$ M1

$\theta \approx 56°$ cao A1 5

Second alternative



$GG' : G'P = \frac{1}{2}M : M = 1 : 2$

$OG = \frac{1}{3}a,\ OP = a$

By similar triangles

$ON = \frac{1}{3}OP = \frac{1}{3}a$ M1 A1

$NG' = \frac{2}{3}OG = \frac{2}{9}a$ A1

$\tan\theta = \frac{ON}{NG'} = \frac{\frac{1}{3}a}{\frac{2}{9}a} = \frac{3}{2}$ M1

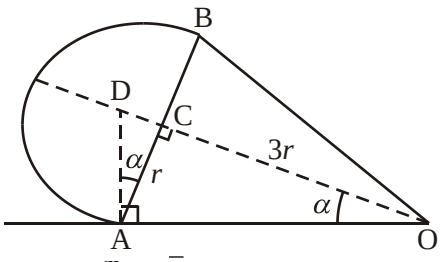
$\theta \approx 56°$ cao A1 5

6. (a) $\frac{3r}{4}$; $\frac{3r}{8}$ B1; B1

$-m.\frac{3r}{4} + M.\frac{3r}{8} = (m+M)\bar{x}$ M1 A1

$\frac{3r(M-2m)}{8(M+m)} = \bar{x}$ (*) A1 5

(b)



No equil$^{m} \Rightarrow \bar{x} \geq CD$ M1

$\frac{3r(M-2m)}{8(M+m)} \geq \frac{r}{3}$ M1 A1

$9(M-2m) > 8(M+m)$

$M > 26m$ (*) A1 c.s.o. 4

[9]

7. (a) $\int_0^{\pi} \frac{1}{2} y^2 \mathrm{d}x = \int_0^{\pi} \frac{1}{2} \sin^2 x \mathrm{d}x$ M1

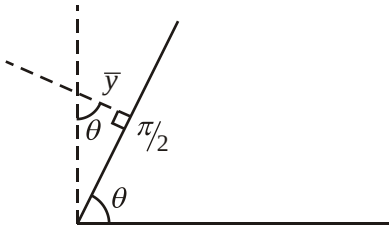
$= \frac{1}{4} \int_0^{\pi} (1 - \cos 2x) \mathrm{d}x$ M1

$= \frac{1}{4} [x - \frac{1}{2} \sin 2x]_0^{\pi}$ A1

$= \frac{\pi}{4}$ A1

$\bar{y} = \frac{\pi/4}{\int_0^{\pi} \sin x \mathrm{d}x} = \frac{\frac{\pi}{4}}{2} = \frac{\pi}{8}$ A1 6

(b)



$\tan\theta = \frac{\pi/2}{\bar{y}}$ M1

= 4 A1ft

θ = 75.96°, 76° A1 3

75.9°, 76.0°

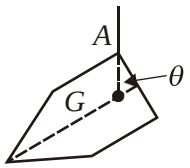
[9]

8. (a)

	Cylinder ($36\pi r^3$)	Cone ($12\pi r^3$)	Toy ($48\pi r^3$)	
mass ratio	3	1	4	B1
dist. from O	$2r$	$(-)r$	$\bar{x}$	B1
		$(3 \times 2r) - r = 4\bar{x}$		M1 A1
		$\frac{5r}{4} = \bar{x}$		A1 5

M1 for clear attempt at $\Sigma mx = \bar{x}\,\Sigma m$ – correct no. of terms.
If distances not measured from O, B1B1M1A1 available.

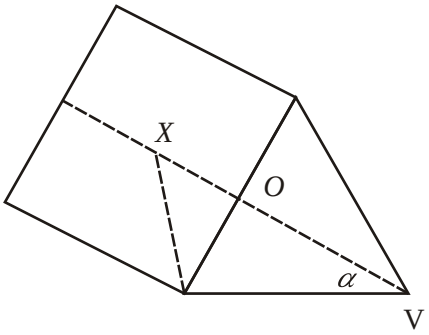
(b)



AG vertical, seen or implied	M1	
$\tan\theta = \dfrac{3r}{4r-\bar{x}}$	M1 A1	
$\theta = 47\ 5°$ (1 d.p.)	A1	4

second M1 for use of tan

(c)



sim Δ's: $\dfrac{OX}{3r} = \dfrac{3r}{4r}$ (= tan α)	M1	
$\Rightarrow OX = \dfrac{9r}{4}$	A1	
$\bar{x} < OX$	M1	
$\Rightarrow$ won't topple	A1 c.s.o	4

Note that second M1 is independent, for the general idea.

[13]

9. (a)

	Cylinder	Hemisphere	S	
Masses	$(\rho)\pi(2a)^2(\frac{3}{2}a)$	$(\rho)\dfrac{2}{3}\pi a^3$	$(\rho)(\frac{16}{3}\pi a^3)$	M1A1
	$[6\pi a^3]$			
	[18]	[2]	[16]	

[M1 for attempt at C, H and S = C – H masses]

Distances of CM from

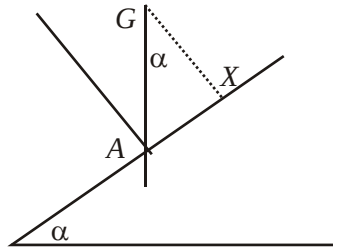
O	$\dfrac{3}{4}a$	$\dfrac{3}{8}a$	$\bar{x}$	B1B1
or lower face	$\dfrac{3}{4}a$	$\dfrac{a}{2}+\dfrac{5a}{8}$	$\bar{x}'$	

Moments equation: $6\pi a^3(¾\ a) - \dfrac{2}{3}\pi a^3(\dfrac{3}{8}a) = \dfrac{16}{3}\pi a^3\ \bar{x}$ M1

$\bar{x} = \dfrac{51}{64}a$ A1 6

$(0.797a)$

(b)



	G above "A" seen or implied	M1	
	or $mg\sin\alpha\,(GX) = mg\cos\alpha\,(AX)$		
	$\tan\alpha = \dfrac{AX}{XG} = \dfrac{2a}{\frac{3}{2}a - \bar{x}}$	M1	
	$[GX = \frac{3}{2}a - \frac{51}{64}a = \frac{45}{64}a$, $\tan\alpha = \dfrac{128}{45}]$ **α = 70.6°**	A1	3

(c)

	Finding F and R : $R = mg\cos\beta$, $F = mg\sin\beta$	M1	
	Using $F = \mu R$ and finding $\tan\beta$ [= 0.8]	M1	
	β = 38.7°	A1	3

[12]

10. (a)

	$V = \pi\int y^2\,dx$ $[= \frac{1}{4}\pi\int (x-2)^4\,dx]$	M1	
	$\int (x-2)^4\,dx = \frac{1}{5}(x-2)^5$ or $[\frac{1}{5}x^5 - 2x^4 + 8x^3 - 16x^2 + 16x]$	M1 A1	
	[M1 requires attempt to square and integrate]		
	$V = \dfrac{8\pi}{5}$	A1	4

(b)

	Using $\pi\int xy^2\,dx$ $= [\frac{1}{4}\pi\int x(x-2)^4\,dx$	M1	
	Correct strategy to integrate [e.g. substitution, expand, by parts]	M1	
	[e.g. $\frac{1}{4}\pi\int (u-2)^4\,du$; $\frac{1}{4}\pi\int (x^5 - 8x^4 + 24x^3 - 32x^2 + 16x)\,dx$];		
	$= \frac{1}{4}\pi\left[\dfrac{2u^5}{5} + \dfrac{u^6}{6}\right]$ or $\frac{1}{4}\pi\left[\dfrac{x^6}{6} - \dfrac{8x^5}{5} + 6x^4 - \dfrac{32x^3}{3} + 8x^2\right]$	M1 A1	
	Limits used correctly [correct values = $\dfrac{8\pi}{15}$]	A1 ft	
	$V_c\,(\rho)\,\bar{x} = \pi(\rho)\int xy^2\,dx$ (seen anywhere)	M1	
	$\bar{x} = \frac{1}{3}$ cm (*) no incorrect working seen	A1	7

(c) Moments equation to find C of M of tool:
e.g. $12W\bar{x} = 10W \times 4 - 2W \times (\frac{1}{3})$ M1 A1 A1

(may be implied by next line) $[\bar{x} = 3\frac{5}{18}$ *from plane edge of S]*

Moments about *B*: $8R_A = 40W - 2W(\frac{1}{3})$

$R_A = \frac{59W}{12}$ (4.9 W or 4.92W) A1 5

[16]

[Moments about other points: M1 A1,
Complete method to find R_A ; using $R_A + R_B = 12W$ with moments equation M1 A1 ft; A1 as scheme]

11. (a)

Cylinder	half-sphere	toy	
$\pi r^2 h\rho$	$\frac{2}{3}\pi r^3 6\rho$	$\pi r^2 h\rho + \frac{2}{3}\pi r^3 6\rho$	M1 A1
$\frac{h}{2} + r$	$\frac{5r}{8}$	d	B1 B1

$\pi r^2 h\rho(\frac{h}{2} + r) + 4\pi r^3\rho \ \frac{5r}{8} = (\pi r^2 h\rho + 4\pi r^3\rho)d$ M1 A1

$\Rightarrow d = \frac{h^2 + 2rh + 5r^2}{2(h+4r)}$ (*) A1 7

(b) $d = r$, $\Rightarrow h^2 + 2rh + 5r^2 = 2r(h + 4r)$ M1, M1

$h = \sqrt{3}r$ A1 3

[10]

1. This was a routine centre of mass problem requiring mass and associated known centre of mass for standard volumes, combined in an appropriate moments equation. A few mistakes occurred when candidates tried to write down their moments equation without detailing each part in a table. However the major difficulty was for those who couldn't produce a correct volume for the cone and occasionally even the cylinder. The cone had multiples of $\frac{2}{3}$ and $\frac{1}{4}$ used with $\pi r^2 h$ and the cylinder became $2\pi r^2 h$. Some candidates failed to introduce the different value for r as l and $2l$ before cancelling it hence giving an incorrect mass ratio.

In part (b), the condition for tipping with G above the bottom point on the container, was used by all who attempted this part. Recognising the use of $6l$ – (their x) was crucial to finding a correct trigonometric ratio, and those that did made few mistakes finding θ correctly. As expected a few used l instead of $2l$ in the numerator for their expression for $\tan\theta$.

2. This question produced many completely correct solutions. Some candidates however ignored the information provided about the masses of the two parts of the route marker and calculated their own "masses" by using volumes instead, usually assuming both sections to be solids. Almost all could produce a valid, if not correct, moments equation and so gained some marks. The most frequently seen error in part (b) was to have the required fraction upside down resulting in h = $\frac{r}{24}$. Some candidates lost the final mark here through giving r in terms of h instead of answering the question asked.

3. This tended to be often an all-or-nothing question. Several gained full marks, but others could not get started. Others too, could find the position of the centre of mass of the cone, but did not appear to realise that this must be vertically below the point A.

4. This provided a very easy nine marks for many and almost all knew what they were trying to do. The most frequent mistake in (a) was to assume that volumes needed to be used for the cylinder and hemisphere and in (b) the fraction was often upside down. Where solutions did go wrong, there was again widespread fiddling. Even the volume versions sometimes ended up being rearranged, factorised or cancelled into the required expression.

5. Part (a) was generally well done and the majority of those who could not obtain the printed answer usually failed through lack of knowledge of the formula for the surface area of the sphere. This formula is given in the Formula Booklet and candidates in Mechanics are expected to be familiar with, or know where to look for, Pure (or Core) Mathematics formulae which are appropriate to a Mechanics module. Part (b) was generally poorly done. Many candidates were unable to visualise the situation and, after a few attempts to draw a diagram, abandoned the question. The simplest solution is to take moments about the point of contact with the floor or about the centre of the plane face of the hemisphere but this was rarely seen. The most successful candidates tended to be those who worked out the position of the centre of mass of the hemisphere and particle combined, even though this involved more work.

6. The first part was generally well done, although some candidates tried to introduce volumes into their calculations but there were very few fully correct solutions to part (b), where a good diagram was essential.

7. The formula required in part (a) was not always well-known and even those that did quote it correctly were not always able to cope with the resulting integral. The second part was totally independent and was generally well-answered.

8. This question was generally very well answered with many fully correct solutions to parts (a) and (b). A pleasing number of candidates also were successful in presenting a logical solution to part (c). Only the weakest candidates found the question inaccessible and scored fewer than half marks. The most common error in part (a) was inconsistent use of r - some candidates being confused with radius as used in the volume formulae and the use of r in the question. This led to inconsistent algebraic equations that proved impossible to simplify and solve. The most common error in part (b) was finding the complement of the required angle or not stating the answer to the required degree of accuracy. Part (c) proved difficult for many candidates - common errors were using 3*r*/2 as the radius of the base of the cone/cylinder and assuming the slant length of the cone to be 4*r*. Correct results for two angles were often calculated but the relationship between them was not sufficiently explained for full marks to be awarded. The most satisfactory solutions involved proving that the line of action of the weight passed through the slant side of the cone or that the sum of the base angle of the cone and the angle from O to the edge of the cone/cylinder interface to G was less than 90 degrees.

9. The vast majority of candidates knew what was required in this question, which was generally a source of high marks.

10. In part (a) the majority of candidates were able to recognise and use the formula for the volume of revolution. The most common approach to finding $\int (x-2)^4 \mathrm{d}x$ was to multiply out the brackets, which will have taken up precious time and was often incorrect.

Problems arose with the nature of the given equation, with many candidates not squaring the squared factor, so that $\frac{1}{4}\pi\int (x-2)^2 \mathrm{d}x$ was often seen, even after $\pi\int y^2 \mathrm{d}x$ had previously been quoted. Also, there were as many candidates who used 0 and 1 as the limits of integration as there were who used the correct values of 0 and 2; presumably triggered by the given statement that "The unit of length on both axes is 1 cm". If this was the only mistake, candidates only lost two marks for this misunderstanding.

Although there were many very good solutions to this question, part (b) did cause problems. It may be that at this stage weaker candidates were rushed and did not read the question well enough as errors such as using $\int xy \,\mathrm{d}x$, not squaring *y*, and poor integration were common.

In part (c) many candidates made the question more unwieldy by working with their own weights rather than using the given weights for *S* and *C*, and it was surprising to see so many candidates finding the centre of mass of the tool on route to answering the question. However, most candidates were able to gain some credit in this part.

11. The correct method was well known in the first part and the majority were able to obtain the printed answer. Common errors were to omit or interchange the densities, use an incorrect volume formula or to measure a distance from a wrong point. There was less success with part (b) where a significant number of candidates did not know where to start.