Problem 1

Consider the Hamiltonian of two particles, of masses m_1 and m_2, interacting via potential $V(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ that depends only on the relative position of the particles:

$$\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + V(\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2). \tag{1}$$

We define the position and the momentum of the center of mass operators

$$\hat{\boldsymbol{R}}_{\text{CM}} := \frac{m_1 \hat{\boldsymbol{r}}_1 + m_2 \hat{\boldsymbol{r}}_2}{m_1 + m_2} \qquad \hat{\boldsymbol{P}}_{\text{CM}} := \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2 \tag{2}$$

and the relative position and the relative momentum operators

$$\hat{\boldsymbol{r}}_{\text{rel}} := \hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2 \qquad \hat{\boldsymbol{r}}_{\text{rel}} := \frac{m_2 \hat{\boldsymbol{p}}_1 - m_1 \hat{\boldsymbol{p}}_2}{m_1 + m_2} \tag{3}$$

a) Show, that the Hamiltonian (1) can be written as the sum of the Hamiltonian for the cener of mass motion and the Hamiltonian for the relative motion $\hat{H} = \hat{H}_{\text{CM}} + \hat{H}_{\text{rel}}$, where

$$\hat{H}_{\text{CM}} = \frac{\hat{P}_{\text{CM}}^2}{2\,(m_1 + m_2)} \qquad \hat{H}_{\text{rel}} = \frac{\hat{p}_{\text{rel}}^2}{2\mu} + V(\hat{\boldsymbol{r}}), \tag{4}$$

where μ is the reduced mass equal to $\frac{m_1\, m_2}{m_1 + m_2}$.

b) Proove the commutation relations

$$\left[\hat{\boldsymbol{R}}_{\text{CM}}, \hat{\boldsymbol{p}}_{\text{rel}}\right] = 0 \tag{5}$$

$$\left[\hat{\boldsymbol{P}}_{\text{CM}}, \hat{\boldsymbol{r}}_{\text{rel}}\right] = 0 \tag{6}$$

c) Show that the commutations relations between the three components of $\hat{\boldsymbol{P}}_{\text{CM}}$ and $\hat{\boldsymbol{R}}_{\text{CM}}$ are the same as for the momentum and the position operator of a single particle.

Problem 2: Spectrum of a particle in the potential $V(r) = -\frac{1}{4\pi\epsilon_0\, r}$

Derive the first approximation of the spectrum of the hydrogen, $E_n = -\frac{E_0}{n^2}$ using your favorite way. Three possible methods of the derivation were sketched during the third part of the lecture 2.

Problem 3: Addition of angular momenta

a) *Addition of J_1 and J_2:* Consider addition of two angular momenta and the case in which $j_1 = 1$ and $j_2 = 1$. Assume the basis composed of common eigenstates of $\mathbf{J}_1^2, \mathbf{J}_2^2, \mathbf{J}_{z1}, \mathbf{J}_{z2}$ to be known, i.e. $|j_1 m_1\rangle|j_2 m_2\rangle$ with $m_1, m_2 = 1, 0, -1$. The task is to determine the $|jm\rangle$ basis of common eigenvectors of $\mathbf{J}^2, \mathbf{J}_1^2, \mathbf{J}_2^2$, where $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ is the total angular momentum.

   Note, that the possible values of the quantum number j are $j = 2, 1, 0$. Construct three families of $|jm\rangle$ vectors each corresponding to the given value of j. The corresponding family will contain five, three and one vectors of the new basis depending on the possible values of m. In each family recognize the state with the maximal value of m. Use the lowering operator $\mathbf{J}_- = \mathbf{J}_{-1} + \mathbf{J}_{-2}$ in order to calculate the states having lower values of m.

b) *Addition of L and S:* Consider an electron bounded with a proton in the state of orbital angular momentum l. Since the electron has spin l, its total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ can have values $j = l \pm 1/2$. Express the total angular momentum states $|jm\rangle$ in terms of product states $|lm_l\rangle|sm_s\rangle$, where m_l, m_s, m stand for orbital, spin and total projections along the z-axis.

   Note, the total angular momentum states $|jm\rangle$ for each m are a superposition of two product states because $m_s = \pm 1/2$. Use the orthonormality condition and the result of the action $\mathbf{J}^2$ on the j-state to find relations between the coefficients that constitute the superposition.

Further reading: Complement A X in "Quantum Mechanics. Volume II: Angular Momentum, Spin, and Approximation Methods", C. Cohen-Tannoudji, B. Diu, F. Laloë

Problem 4: Determination of the Clebsch-Gordan coefficients

The total angular momentum states $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ can be expressed in terms of product states

$$|jm\rangle = \sum_{m_1, m_2} C^{jm}_{j_1 m_1; j_2 m_2} |j_1 m_1\rangle|j_2 m_2\rangle, \tag{7}$$

where $C^{jm}_{j_1 m_1; j_2 m_2}$ are the Clebsch-Gordan coefficients.

a) Show that the Clebsch-Gordan coefficients $C^{jm}_{j_1 m_1; j_2 m_2}$ are non-zero only if $m = m_1 + m_2$.

b) Using $J_+ = J_{1+} + J_{2+}$ prove the following recursion relation

$$\sqrt{j(j+1) - m(m-1)}C^{jm+1}_{j_1 m_1; j_2 m_2} = \\ \sqrt{j_1(j_1+1) - m_1(m_1-1)}C^{jm}_{j_1 m_1 - 1; j_2 m_2} + \sqrt{j_2(j_2+1) - m_2(m_2-1)}C^{jm}_{j_1 m_1; j_2 m_2 - 1} \tag{8}$$

Deduce from this a relation between $C^{jm}_{j_1 m_1 - 1; j_2 m_2}$ and $C^{jm}_{j_1 m_1; j_2 m_2 - 1}$.