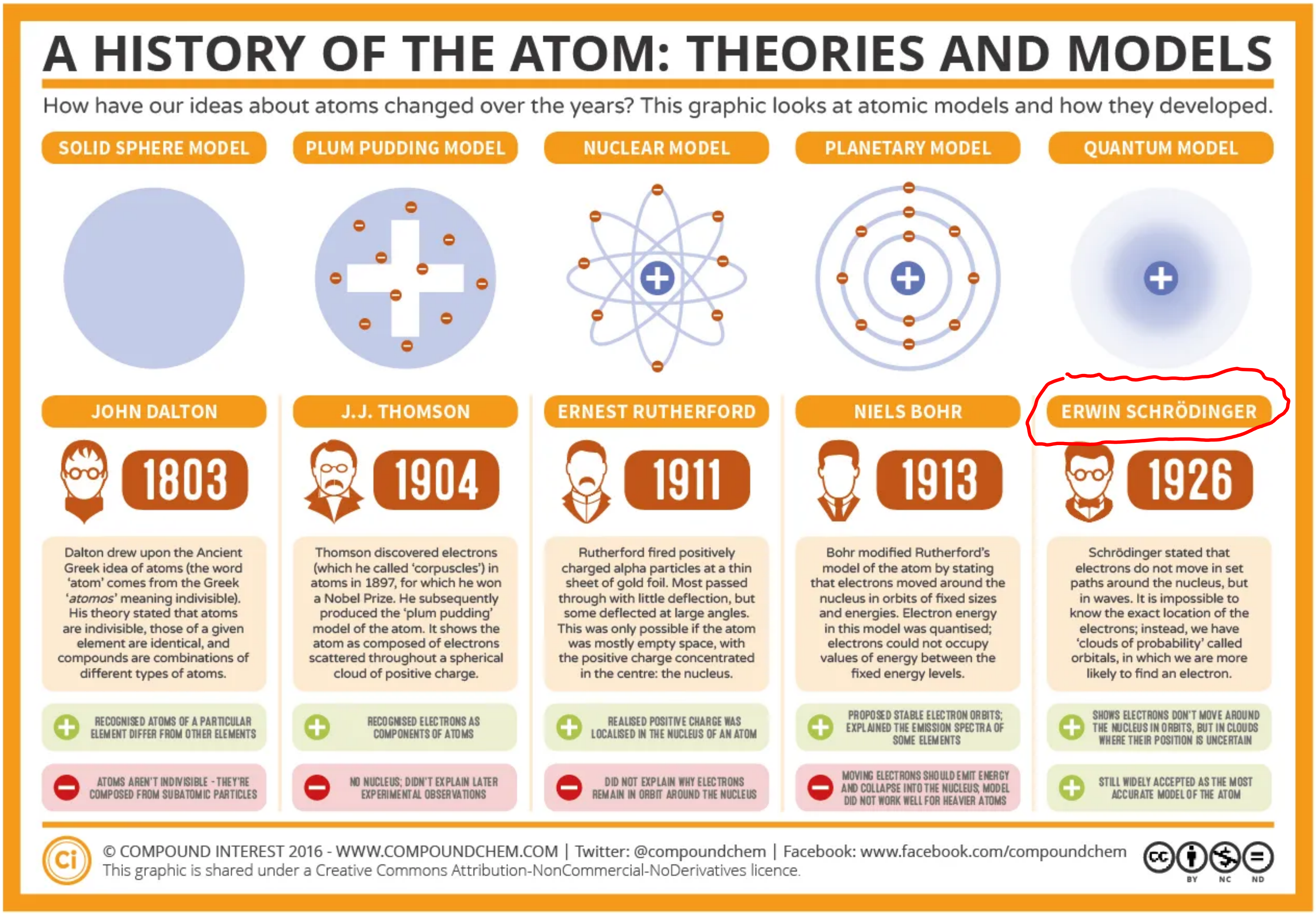
Lecture 2 Warsaw, 28/10/2020

Atomic and Optical Physics I

https://ocw.mit.edu/courses/physics/8-421-atomic-and-optical-physics-i-spring-2014/

Last time: history of an atom and classical models of an atom



How do we know atoms exist ?

The first direct observation of the orbital structure of an excited hydrogen atom

→ https://physicsworld.com/a/quantum-microscope-peers-into-the-hydrogen-atom/

What are atoms made off ?

mass of 1836 electrons = mass of 1 proton;

mass of 1 neutron = mass of 1 proton (more or less)

Atoms have as many elektrons as protons.
Atoms usually have as many neutrons as protons.

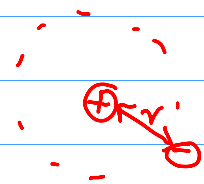
Adding a proton makes a new kind of atom!

Adding a neutron makes an isotope of atom a heavier version of that atom !

Is it enough to put close to each other some of these elements to construct and understand, characterize the atom?

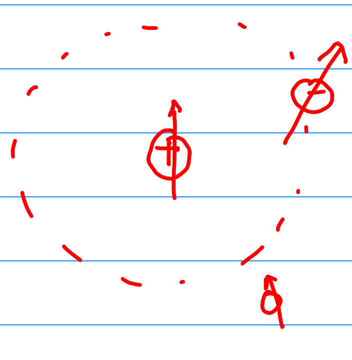
Interactions: Coulomb interactions between electron and proton

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$$



Interactions: take into account that these particles have a $\boxed{\text{spin}}$ and
the orbital motion of electrons induce importance of angular momentum

S - spin



$L = \vec{r} \times \vec{p}$ - angular momentum

Spin: formalism

$\vec{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ $[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$

$s = \frac{\hbar}{2}$

$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $S_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$s = \hbar$

$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

$S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z$ $[\hat{S}^2, \hat{S}_z] = 0$

$\hat{S}^2 |s, s_z\rangle = s(s+1) |s, s_z\rangle$ $\hat{S}_z |s, s_z\rangle = s_z |s, s_z\rangle$

Hilbert space

V_0 – orbital degrees of freedom
V_s – spin degrees of freedom

$V = V_0 \otimes V_s$

$H = H_0 + H_s$ $|\psi\rangle = |\psi_0\rangle \otimes \boxed{|\chi_s\rangle}$

$|\chi_s\rangle \overset{s=\hbar/2}{=} \begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Problems:

Consider particle with spin 1,

$$H = H_0 + \alpha S_z \qquad [H_0, S_x] = [H_0, S_y] = [H_0, S_z] = 0$$

$$t=0; \quad |\psi(0)\rangle = |z\rangle \qquad \hat{S}_z |z\rangle = \hbar |z\rangle$$

A) $\langle \hat{S}_z \rangle$, what is the probability to obtain $\pm\hbar$?

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad |S_z - \lambda \mathbb{1}| = \begin{vmatrix} \hbar-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\hbar-\lambda \end{vmatrix} = +\lambda(\hbar^2 - \lambda^2) = 0$$

eigenvalues: $\lambda_1 = 0$; $\lambda_2 = \hbar$, $\lambda_3 = -\hbar$

eigenvectors: $(S_z - \lambda_i \mathbb{1}) z_i = 0$

$\lambda_1 = 0$

$$\begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \hbar a \\ 0 \\ -\hbar c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad a = c = 0, \; b - \text{any} \qquad z_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda_2 = \hbar$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\hbar & 0 \\ 0 & 0 & -2\hbar \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -\hbar b \\ -2\hbar c \end{pmatrix} = 0 \Rightarrow \quad \boxed{z_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$\lambda_3 = -\hbar$

$$\begin{pmatrix} 2\hbar & 0 & 0 \\ 0 & \hbar & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2\hbar a \\ \hbar b \\ 0 \end{pmatrix} \Rightarrow \quad z_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|\psi(T)\rangle = e^{-itH/\hbar} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^{-i(H_0 + \alpha S_z)t/\hbar} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$e^{A+B} = e^{-[A,B]/2} e^{A} e^{B} \qquad [H_0, S_z] = 0$$

$$\hookrightarrow |\bar{\psi}(t)\rangle = e^{-iH_0t/\hbar}\, e^{-i\mathcal{L}S_z t/\hbar}\begin{pmatrix}1\\0\\0\end{pmatrix} = e^{-iH_0t/\hbar}\times$$

$$\times \sum_n \left(\frac{-i\mathcal{L}t}{\hbar}\right)^n \frac{1}{n!} S_z^n \begin{pmatrix}1\\0\\0\end{pmatrix} = e^{-iHt/\hbar}\sum_n \left(-\frac{i\mathcal{L}t}{\hbar}\right)^n \hbar^n \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$= e^{-iH_0t/\hbar}\, e^{-i\mathcal{L}t}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$[H_0, S_z] = 0$$

$$H_0|z_i\rangle = \mathcal{E}_0|z_i\rangle$$

$$|\psi(t)\rangle = e^{-i\mathcal{E}_0 t}\, e^{-i\mathcal{L}t}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$P_\hbar = |\langle z_2|\psi(t)\rangle|^2 = \left|(1\,0\,0)\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot e^{-i\mathcal{E}_0t}e^{-i\mathcal{L}t}\right|^2 = 1$$

$$P_{-\hbar} = |\langle z_3|\psi(t)\rangle|^2 = \left|(0\,0\,1)\begin{pmatrix}1\\0\\0\end{pmatrix}\right|^2 = 0$$

$$P_0 = 0$$

(B) if we will $\langle S_x\rangle$; what is $P_{\pm\hbar}$?

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad |S_x - \lambda \mathbb{1}| = \begin{vmatrix}-\lambda & \frac{\hbar}{\sqrt2} & 0\\ \frac{\hbar}{\sqrt2} & -\lambda & \frac{\hbar}{\sqrt2}\\ 0 & \frac{\hbar}{\sqrt2} & -\lambda\end{vmatrix} =$$

$$= -\lambda^3 + 0 + 0 - 2(-\lambda)\left(\frac{\hbar}{\sqrt2}\right)^2 = -\lambda^3 + \lambda\hbar^2 =$$

$$= \lambda(\hbar^2 - \lambda^2)$$

$$\lambda_1 = 0 \,; \quad \lambda_2 = \hbar \,; \quad \lambda_3 = -\hbar$$

$$x_1 = \frac{1}{\sqrt2}\begin{pmatrix}1\\0\\-1\end{pmatrix} \qquad x_2 = \frac12\begin{pmatrix}1\\\sqrt2\\1\end{pmatrix} \qquad x_3 = \frac12\begin{pmatrix}1\\-\sqrt2\\1\end{pmatrix}$$

$$P_{\hbar} = |\langle x_2 | \psi(t)\rangle|^2 = \left|\frac{1}{2}(1, \sqrt{2}, 1)\begin{pmatrix}1\\0\\0\end{pmatrix} e^{-i\varepsilon_0 t} e^{-i\mathcal{L}t}\right|^2 =$$

$$= \frac{1}{4}$$

$$P_{-\hbar} = |\langle x_3 | \psi(t)\rangle|^2 = \left|\frac{1}{2}(1, -\sqrt{2}, 1)\begin{pmatrix}1\\0\\0\end{pmatrix}\right|^2 = \frac{1}{4}$$

$$P_0 = |\langle x_1 | \psi(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1, 0, -1)\begin{pmatrix}1\\0\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

(c) Let assume that at $t=T$, $\langle S_x\rangle$ is measured $-\hbar$ is obtained; and after at $\tau > T$ once again we measure $\langle S_x\rangle$. What is $P_{0,\pm\hbar}$?

$$|\psi(T)\rangle = |x_3\rangle = \frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$$

$$|\psi(t>T)\rangle = e^{-i(H_0 + \mathcal{L}S_z)t/\hbar} \cdot |x_3\rangle = e^{-iH_0 t/\hbar} e^{-i\mathcal{L}S_z t/\hbar} |x_3\rangle$$

$$|x_3\rangle = \frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix} = \frac{1}{2}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix} - \sqrt{2}\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} =$$

$$= \frac{1}{2}(z_1 - \sqrt{2} z_2 + z_3)$$

$$|\psi(t>T)\rangle = \frac{1}{2} e^{-iH_0 t/\hbar} e^{-i\mathcal{L}S_z t/\hbar}(z_1 - \sqrt{2} z_2 + z_3) =$$

$$= \frac{1}{2} e^{-iH_0 t/\hbar}\left(e^{-i\mathcal{L}t} z_1 - \sqrt{2} z_2 + e^{+i\mathcal{L}t} z_3\right)$$

$$= \frac{1}{2} e^{-i\varepsilon_0 t/\hbar}\left(e^{-i\mathcal{L}t} z_1 - \sqrt{2} z_2 + e^{i\mathcal{L}t} z_3\right)$$

$$P_0 = |\langle x_1 | \psi(t>T)\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1, 0, -1)\frac{1}{2}\left\{e^{-i\mathcal{L}t}\begin{pmatrix}1\\0\\0\end{pmatrix} - \sqrt{2}\begin{pmatrix}0\\1\\0\end{pmatrix} + e^{i\mathcal{L}t}\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\right|^2$$

$$= \frac{1}{2}\cdot\frac{1}{4}\left| e^{-i\mathcal{L}t} - e^{-i\mathcal{L}t}\right|^2 = \frac{1}{2}\cdot\frac{1}{4}\cdot|2i\sin\mathcal{L}t|^2 =$$

$$= \frac{1}{2}\sin^2\mathcal{L}t$$

$$P_{\hbar} = |\langle x_2 | \psi(t>T) \rangle|^2 = \left| \tfrac{1}{2}(1, \sqrt{2}, 1) \tfrac{1}{2} \left\{ e^{-i\Omega t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \sqrt{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^{i\Omega t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \right|^2 =$$

$$= \frac{1}{4} \cdot \frac{1}{4} \left| e^{-i\Omega t} - 2 + e^{i\Omega t} \right|^2 = \frac{1}{4 \cdot 4} \cdot \left| 2 \cos \Omega t - 2 \right|^2 =$$

$$= \frac{1}{4} (1 - \cos \Omega t)^2$$

$$P_{-\hbar} = \frac{1}{4} (1 + \cos \Omega t)^2$$

$P_{\hbar} = 0 \; ; \quad P_{-\hbar} = 1$ $\quad \cos \Omega t = 1 \quad \sin \Omega t = 0$

Next on-line lecture // meeting

4th of October!

Problems to do at home!