Problem 1

Prove that in motion under the force of $\boldsymbol{F}(\boldsymbol{r}) = -k\boldsymbol{r}/r^3$, the Runge-Lentz vector $\mathbf{A}$ is conserved as in the planetary model of an atom,

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - km\frac{\mathbf{r}}{r}. \tag{1}$$

See lecture 1b for hints.

Problem 2: Meaning of uncertainty relation

The ground state of the 1D harmonic oscillator in position representation reads:

$$\psi_{\text{GS}}(x) = \frac{1}{(\pi a^2)^{1/4}} e^{-x^2/(2a^2)}, \tag{2}$$

where $a = \sqrt{\hbar/(m\omega)}$ is the oscillatory length. In the momentum representation, the same state reads

$$\tilde{\psi}_{\text{GS}}(p) = \frac{1}{(\pi\hbar m\omega)^{1/4}} e^{-p^2/(2\hbar m\omega)}, \tag{3}$$

a) Simulate $10 + 10$ measurements: use any programming language/Mathematica to draw $k = 10$ positions from the distribution $|\psi_{\text{GS}}(x)|^2$ and $k = 10$ momenta from the distribution $|\tilde{\psi}_{\text{GS}}(p)|^2$.

b) Estimate averages and dispersions:

$$\bar{x} = \tfrac{1}{k}\textstyle\sum_{i=1}^{k} x_i \qquad \bar{x^2} = \tfrac{1}{k}\sum_{i=1}^{k} x_i^2 \tag{4}$$

$$\bar{p} = \tfrac{1}{k}\textstyle\sum_{i=1}^{k} p_i \qquad \bar{p^2} = \tfrac{1}{k}\sum_{i=1}^{k} p_i^2 \tag{5}$$

$$\Delta_{\text{est}}x := \sqrt{\bar{x^2} - \bar{x}^2} \qquad \Delta_{\text{est}}p := \sqrt{\bar{p^2} - \bar{p}^2} \tag{6}$$

What is the value of the product $\Delta_{\text{est}}x\ \Delta_{\text{est}}p$ compared to $\hbar/2$?

c) Repeat a) and b) $n = 10000$ times. How often $\Delta_{\text{est}}x\ \Delta_{\text{est}}p < \hbar/2$?

d) Repeat a), b) and c), but this time "measuring $k = 100$ positions and $k = 100$ momenta.

In the answer give graphical presentation of results for b), c) and d), with well described captions.

Problem 3: Reminder of the quantum harmonic oscillator

Show that for eigenstates of harmonic potential an average value of kinetic energy equals an average value of potential energy, $\langle E_{kin} \rangle = \langle E_{pot} \rangle$.

Problem 4: Reminder of the box potential

Find the bound states of a particle in an one-dimensional box, for which the potential is

$$V(x) = -V_0 \quad \text{for } x \in [-L, L], \tag{7}$$

and $V(x) = 0$ elsewere, where $V_0 > 0$. Hints: Solve the Schrödinger equation in the three regions requiring $\psi(x) \to 0$ for $x \to \pm\infty$, continuous $\psi(x)$ and $\psi'(x)$ at $x = \pm L$. Display the eigenvalues spectrum for $E < 0$, and discuss its dependence on L and V_0.