Approximation of Inflation-Adjusted Returns

Yaman Kındap, Arman Boyacı

September 21, 2020

In this work, we analyzed the Fisher equation which is an approximation for inflation-adjusted returns. Our analysis shows that the approximation is valid when the values of inflation and returns are close to zero and both values are close to each other. The error made by using the approximation is visualized in order to provide an understanding of the effect of inflation on real returns.

Introduction

The relationship between the returns of an investor and the change in her purchasing power is non-trivial. The overall change in the price levels of available goods should also be considered when we are measuring the performance of an investment or comparing multiple potential global portfolios. Let's denote the return of an investment as r and the rate of inflation in our currency as i. Then the rate of return in proportion to the inflation can be calculated as:

$$r_{adj} = \frac{1+r}{1+i} - 1 \tag{1}$$

where r_{adj} is the inflation adjusted rate of return.

The Fisher equation [1] shows that under small values of both r and i Equation (1) can be approximated as:

$$r_{adj} \approx r - i \tag{2}$$

While equation 1 is easy to compute in a computer, the approximate equation 2 enables us to make quick back-of-the-envelope calculations. We find that the error made by using the approximate equation has certain patterns in different inflationary environments. The derivation of the approximate equation and the error made by using this equation are discussed in the coming sections.

[1] named after Irving Fisher, an American economist, made contributions to utility theory. Not to confuse with Ronald Fisher who is a British statistician known for Fisher information.

Derivation of the Approximation

The approximation of the inflation adjusted rate of return depends on the Taylor series expansion [2] of a function around a single point a. In order to transform the division in Equation (1), we can take the logarithm of both sides. This yields:

$$\log(1 + r_{adj}) = \log(1 + r) - \log(1 + i) \tag{4}$$

[2] Formally, the Taylor series of a function $f(x)$ is shown as:

$$f(x) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{3}$$

where $f^{(n)}(a)$ denotes the n^{th} derivative of f evaluated at the point a. Notice that this approximation is only useful when x is close to a.

Let's use $f(x) = \log(1+x)$ and approximate this function around 0 using the Taylor series expansion. The first order approximation of $f(x)$ can be shown as:

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) \tag{5}$$

The first derivative of $\log(1+x)$ is:

$$\frac{\partial}{\partial x}\log(1+x) = \frac{1}{1+x} \tag{6}$$

Thus, the first order approximation of $\log(1+x)$ around 0 is:

$$\begin{aligned} \log(1+x) &\approx \log(1) + \frac{1}{1+0}(x-0) \\ &= x \end{aligned} \tag{7}$$

Hence, assuming that both r and i are sufficiently close to 0, the following holds:

$$\log(1 + r_{adj}) \approx r - i \tag{8}$$

In order to obtain an expression for r_{adj}, we can take the exponential of $\log(1 + r_{adj})$:

$$1 + r_{adj} \approx e^{(r-i)} \tag{9}$$

We can also get rid of the exponential in the right hand side of the equation by using the Taylor series approximation of the exponential function. The first order approximation of e^x around 0 is:

$$\begin{aligned} e^x &\approx e^0 + e^0(x-0) \\ &= 1 + x \end{aligned} \tag{10}$$

Using the Taylor series expansion of the exponential function on equation 9, we obtain:

$$\begin{aligned} r_{adj} &\approx 1 + (r-i) - 1 \\ &= r - i \end{aligned} \tag{11}$$

Notice that we have made three assumptions to arrive at the approximate inflation adjusted return: (1) r is close to 0, (2) i is close to 0 and (3) $r - i$ is close to 0. Thus, only under these special circumstances the approximation may be used.

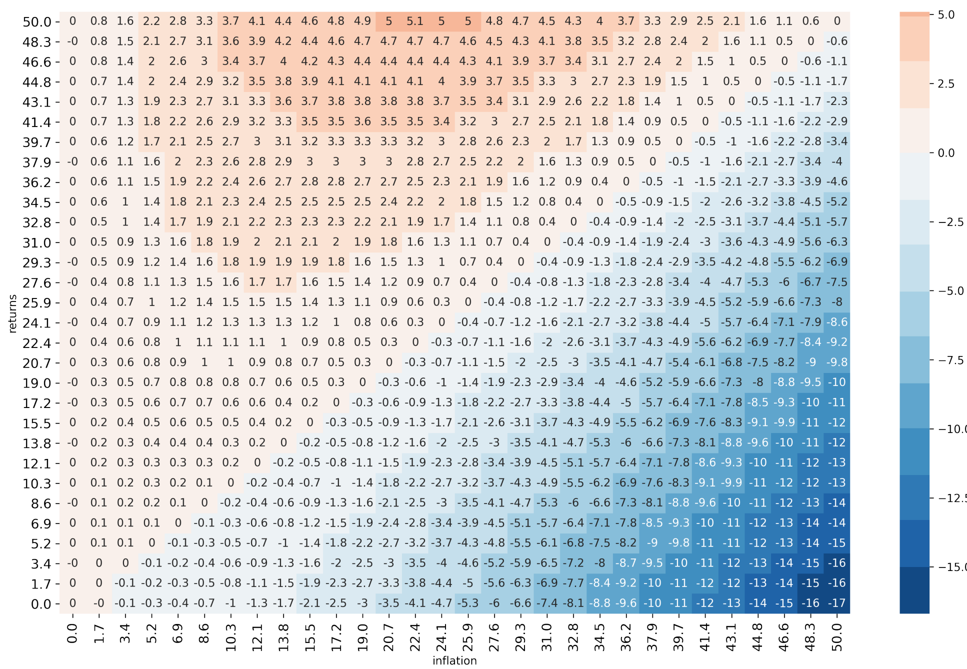

Figure 1: Visualization of approximation errors.

Analysis of Approximation Error

In Figure 1, we provide a visualization of how the approximation error [3] of inflation adjusted returns is changing depending on i and r. Let's first note that the relationship between the approximation error and returns are nonlinear. Our observations of the error space is discussed below:

- The largest error is encountered for small inflation values and large returns as shown in the bottom right corner of the figure.
- The error has a large negative value which means the inflation adjusted returns are underestimated. On the other hand for large inflation values and small returns, as shown in the top left corner of the figure, the error has a small positive value which means that the inflation adjusted returns are overestimated. In reality these edge cases will be rarely encountered.
- Notice that for the case that the inflation and return values are equal and the inflation adjusted returns are equal to zero, the approximation error is always equal to zero. This fact can be easily checked by plugging the numbers to equation 12.

[3] The approximation error is defined as:

$$\text{error} = (r - i) - \left(\frac{1+r}{1+i} - 1\right) \tag{12}$$

Errors are usually computed in L1 or L2-norms. But in this case we wanted to show under (and over) estimation.

- Error increases as the difference between inflation and return values increase. This fact is intuitive since the derivation of the approximate inflation adjusted returns explicitly assumes that the difference in two values is close to zero.
- Figure 1 can be easily divided into two triangles according to the values of axes. The approximation always overestimates the inflation adjusted gains when the returns are greater than inflation and it always underestimates the inflation adjusted losses when the returns are less than inflation.