Cooperative Game Theory - From Basics to Continuums

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1 Introduction

1.1 Personal Background

In the summer of 2015, I had the opportunity to attend the International Summer Program in Economics Education at the Hebrew University of Jerusalem. Being an Economics program in Israel, many of my Professors were game theorists. Now more mathematically mature, and given the topic of this class, my personal motivation for pursuing a topic in game theory was to follow-up on a old self-promise to eventually properly study the subject. The decision to pursue cooperative game theory in particular was because many of the pioneers of the subject have also spent time at Centre for Rationality at the HebrewU.

1.2 Modelling Motivation

As for the importance of the topic of cooperative game theory as a whole, hopefully the mathematical importance is implicit in this project's presentation. As for practical importance, we give three examples.

First, consider the fact (or, at least sentiment) that Canadian phone carriers charge more than their US counterparts. Why is this so and how can we possible equalize things? Finite cooperative game theory provides the flexibility of modelling this situation as the agents being different phone carriers, who may each form a coalition to fix the price. In a more general sense, game theory can aid in the modelling of how oligopolies turn into cartels, and how one can dismantle a cartel.

Second, consider a common shareholder distribution for a particular company where several individuals own most of the company's stock, and then there is a plethora of others who own a small portion. If each shareholder is assumed to have a voting power on company's management direction, then it may be fruitful to model the voting mechanism as an approximation of a coalition game with a measure with atomic and non-atomic values for those who own most of the company's stock vs a small portion respectively.

Finally, consider the example of voting in general. One single voting for one candidate or another is quite insignificant. However, people may form coalitions because if there are three candidates, and one is in their eyes despicable, they could agree to vote for the one with higher perceived probability of winning despite their preference for the other two candidates. Thus modelling voting systems via a coalition game with a continuum of players may be fruitful.

1.3 Summary

The first section of this text focuses on the basics of cooperative game theory: finite coalition games. Our presentation of the subject introduces new notation and definitions deviating from standard so that we may introduce the idea of a restricted coalition game which has the ability to standardize the presentations of solutions concepts.

We then give a small introduction into some important spaces in infinite coalition games. We then conclude this section with a Theorem by Aumann & Shapley on values one of the introduced spaces.

2 Finite Coalition Games

2.1 Preliminary Definitions

Our treatment of finite coalition games is inspired by the desire to explain several stability conditions for games, (or, equivalently, classes of solutions) which still play prominent roles in applications. Our treatment of this subject was chiefly inspired from Chapters 13 & 14 in [4]. However, beside the first few definitions, our notation, definitions, and approach differ substantially. We include proofs, examples, or further explanations based on whichever may best illustrate the concept. We therefore begin with the preliminary definitions:

Definition. 2.1: (Finite) Coalition Game With Transferable Payoff

We denote a coalition game with transferable payoff by the tuple $\langle N, v\rangle$, where $N = \{1, \ldots, n\}$ is considered a "set of players", and a "worth function", $v : 2^N \to \mathbb{R}$ such that $v(\varnothing) = 0$.
$S \in 2^N$ may therefore be thought of as a coalition, and $v(S)$ as that coalition's worth.

A note: a less restrictive definition would exclude the transferable payoff requirement. To do so, we keep the set N as before, but relax our definition of a "worth function" to a function $V : 2^N \to X$, where X is considered a "set of consequences". To then (hopefully) make the game well posed, we also assign to each player $i \in N$ a preference relation $\succsim_i$ on X. While we can consider a coalition game with transferable payoff as a simple coalition game without transferable payoff, our treatment of the topic doesn't necessitate this, and so we henceforth just consider a coalition game as by definition having transferable payoff.

As for a concrete example of a game with transferable utility, consider the following game:

Example. 2.1: N-Player Majority

Given $N = \{1, \ldots, n\}$, then players $1, \ldots, n$ can either divide a unit of payoff among themselves: $v(N) = 1$. Further, for any $S \subset N$ with $|S| = n - 1$, they may divide $\alpha \in [0, 1]$ units of payoff: $v(S) = \alpha$. Finally, for any $|S| < n - 1$, $v(S) = 0$.

Given Example 2.1, we motivate the next definition by the necessity to characterize the division of payouts possible:

Definition. 2.2: Feasible Payouts

If $S \subset N$ and $x(S) \in \mathbb{R}^{|N|}$ such that $\sum_{j=1}^{|N|} x(S)_j = v(S)$ and $x(S)_j = 0 \ \forall j \notin S$, then we say that $x(S)$ is a feasible payoff for the coalition S. We let $X(S)$ denote the set of all possible feasible payoffs for a coalition S. We therefore have:

$$X := \left\{ \sum_{j=1}^{k} x(S_j) \ : \ \mathbb{S} := \{S_1, \ldots, S_k\} \text{ a partition of } N,\ x(S_j) \in X(S_j) \right\} \subset \mathbb{R}^{|N|}$$

as the set of all feasible payoffs for $\langle N, v\rangle$. Finally, if $\mathbf{x} \in X$, we let $\mathbf{x}_i$ represent the i'th coordinate of $\mathbf{x}$ - I.e., agent i's payoff under $\mathbf{x}$ (with coalitions $\{S_1, \ldots, S_k\}$ formed). Thus $x(S) \subset X(S) \subset X \ \forall S \subset N$.

Therefore, by looking at the set of all feasible payoffs, X, for each agent $i \in N$ it induces a preference ordering. Explicitly, if $\mathbf{x}, \mathbf{y} \in X$, then $\mathbf{x} \succsim_i \mathbf{y} \iff \mathbf{x}_i \geq \mathbf{y}_i$. The next subsection is devoted to clarifying the following statement: A feasible payoff is a solution if each agent $i \in N$ cannot leave his current coalition to obtain more "worth" whilst convincing other agents to form a coalition with him, and the new coalition that i formed will not have any other agent leave or be persuaded to leave that new coalition by someone else.

Returning to Example 2.1, if $S = N$, we see that $X(N) =$ the standard $(n-1)$-simplex: $\{\mathbf{x} \in \mathbb{R}^{|N|} : \sum_{i=1}^{|N|} \mathbf{x}_i = 1 \}$, and $X(S_i) = X(N \backslash \{i\})$ is the scaled-by-α $(n-2)$ simplex in $\mathbb{R}^{|N|}$ with a zero in the i'th coordinate.

2.2 An Aside on Finite Stability / Solutions

Given a coalition game, $\langle N, v\rangle$, the interpretation of a solution to said game may correctly be interpreted as: for which feasible payoffs $\mathbf{x} \in X$ will $\mathbf{x}$ be stable in that each agent $i \in N$ cannot increase their own payoff, $\mathbf{x}_i$. But, as intended, the statement "cannot increase their own payoff" is not rigorously defined for this is quite a nuanced question due to its complexity in general.

For example, consider an arbitrary feasible payoff $\mathbf{x} \in X$ associated with the coalitions $\mathbb{S} = \{S_1, \ldots, S_k\}$. Then if $S'_1 \notin \{S_1, \ldots, S_k\}$, it's *possible* that the coalition S'_1 will form (thereby negating the feasibility of $\mathbf{x}$), and decide upon it's feasible payoff $x(S'_1)$ if one of the two conditions hold:

1. $x(S'_1)_i > \mathbf{x}_i$
2. Given that a partition of $N \backslash S'_1$ is created: $\{S'_2, \ldots, S'_{k'}\} =: \mathbb{S}' \backslash S'_1$, and each of these coalitions decide upon a feasible payoff $x(S'_j)$, there may exist a further coalition $S''_1 \notin \mathbb{S} \cup \mathbb{S}'$ for which $i \in S''_1$ will have a decided upon feasible payoff $x(S''_1)$ for which $x(S''_1)_i > \mathbf{x}_i$.

This, however, is already an over-simplification since agent i is only looking 2 periods of "re-coalitioning" into the future. In general, an agent may accept being apart of a coalition so long as he expects to eventually be a part of a coalition which gives him payoff $> \mathbf{x}_i$. Yet even with that in view, it's an over-simplification since every agent is strategizing on how to account for all the subsequent actions other agents would make given that it accepts one temporary coalition over another. The complexity of this problem is immense. This therefore motivates our next definition:

Definition. 2.3: Coalition Game Restrictions

For a coalition game $\langle N, v\rangle$, we say that a set of equalities and inequalities, R, on set of feasible payoffs X is restriction or simplification.

Our definition is derived from the following reasoning: If we consider only a subset of allowable feasible payoffs, then we are finding a solution before saying what type of agent behaviour would result in such a solution - therefore the exact coalition game being played may be inferred by the restrictions on X.

Finite coalition game theory therefore has the form of studying different restrictions and performing analysis on the different types of restrictions possible.

2.3 Restricted Coalition Games

We now consider 3 different restrictions for coalition games.

2.3.1 The Core

Definition. 2.4: The Core's Restriction

We say that a coalition game $\langle N, v\rangle$ has the The Core's Restriction if $\forall \mathbf{x} \in X$, $v(S) \leq \sum_{j \in S} \mathbf{x}_j$ $\forall S \subset N$.

We continue with our Example 2.1 by defining the solution of that coalition game under this forward restriction. In particular, trivially no coalition will form for a coalition S such that $|S| < n - 1$. For a coalition, suppose WLOG that $S = N \backslash \{i\}$ and has payoff $\mathbf{x}$. Since $\sum_{j \in S} \mathbf{x}_j = v(S)$, there must exist $k \in N$ s.t. $\mathbf{x}_k > 0$. If we form the coalition $S' = N \backslash \{k\}$, s.t. $\mathbf{x}'_j = \mathbf{x}_j + \mathbf{x}_k/(n-1)$ for all $j \in S'$ and $\mathbf{x}'_k = 0$, then $\mathbf{x}'$ is a feasible payoff s.t. $\mathbf{x}'_j > \mathbf{x}_j$ $\forall j \in S'$. Thus no coalition S with $|S| < n$ will result in a solution $\mathbf{x} \in X$.

However, consider $S = N$, then for an arbitrary $\mathbf{x} \in X$ s.t. $\mathbf{x}_i > 0$ $\forall i \in N$, it will only be preferential to form a new coalition $S' = N \backslash \{i\} \iff (\sum_{j \in S'} \mathbf{x}_j) < \alpha$ by definition of the restriction. The interpretation is: if $\sum_{j\in S'} \mathbf{x}_j) < \alpha$, then define $\mathbf{x}'_j = \mathbf{x}_j + \big(\sum_{j\in S'} \mathbf{x}_j - \alpha\big)/(n-1) > \mathbf{x}_j$ for all $j \in S'$ and $\mathbf{x}_i = 0$. Thus, the core can be stated explicitly as the subset of X such that $\sum_j \mathbf{x}_j = 1$ and $\sum_{N\setminus\{i\}} \mathbf{x}_j \geq \alpha \ \forall i \in N$.

In general, one may equivalently think of the inequality of "$v(S) \leq \sum_{j\in S} \mathbf{x}_j$" as colloquially meaning there are no deviations such that when the coalition S acts independently, they make more than when partnering with their current coalition.

2.3.2 The Stable Sets (Solutions) of von Neumann & Morgenstern

Definition. 2.5: The von Neumann-Morgenstern Inverse Restriction

We say that a coalition game $\langle N, v\rangle$ has the von Neumann-Morgenstern Inverse Restriction if, for the set of feasible payoffs $\mathbf{x} \in X$, the following restrictions hold:

1. $\mathbf{x}_i \geq v(\{i\})$. (Such feasible payoffs are usually called imputations.)
2. $\exists Y \subset X$ such that $\mathbf{x} \in Y$ and:
   (a) $\forall \mathbf{y} \in Y$, then $\forall S \subset N$, $\nexists\, \mathbf{y}' \in Y$ such that $\sum_{j\in S} \mathbf{y}'_j > \sum_{j\in S} \mathbf{y}_j$. (Internal stability.)
   (b) $\forall \mathbf{z} \in X\setminus Y$, $\exists \mathbf{y} \in Y$ and a $S \subset N$ such that $\sum_{j\in S} \mathbf{y}_j > \sum_{j\in S} \mathbf{z}_j$ (External stability.)

Y's satisfying (1) and (2) are referred to as a von Neumann-Morgenstern Stable Set.

Continuing with our running Example 2.1, consider the case of $\alpha = 1$ and $n = 3$. Then we claim that $Y = \{(1/2, 1/2, 0), (1/2, 0, 1/2), (0, 1/2, 1/2)\}$ is a von Neumann-Morgenstern Stable Set. Condition 1 is trivially satisfied since $v(\{i\}) = 0 \ \forall i \in N$. Condition 2a) is satisfied since if $\mathbf{y}_i \in Y$, then $\mathbf{y}_j$ is not possible since we don't have strict inequalities for both players of any new coalition S'. As for 2b), if $\mathbf{x} \in X$, since $\sum_i \mathbf{x}_i = 1$, $\exists \mathbf{x}_i < 1/(n-1)$ $\exists i, k \in N$ s.t. for $S' = \{i, k\}$, $1 > \sum_{k\in S'} \mathbf{x}_k$.

The colloquial interpretation of this solution is that we are defining stability in the sense that upon a viable perturbation from the current set of implemented coalitions, we will return to another set of coalitions less preferable to the original coalitions.

2.3.3 The Shapley Value

Definition. 2.6: The Shapley Restriction

We say that a coalition games $\langle N, v\rangle$ has the Shapley Restriction if for the set of feasible payoffs $\mathbf{x} \in X \equiv X(v)$, the following restrictions hold:

1. If $v(S \cup \{i\}) = v(S \cup \{j\}) \ \forall S \subset N$ for which $i, j \notin S$, then $\mathbf{x}_i = \mathbf{x}_j$
2. If $v(S \cup \{i\}) - v(S) = v(\{i\}) \forall S \subset N, \{i\} \notin S$, then $\mathbf{x}_i = v(\{i\})$.
3. If $\langle N, w\rangle$ another coalition game with feasible payoffs $X(w)$, then defining a third game by $\langle N, u\rangle := \langle N, v + w\rangle$ with coalition worth's $u(S) = v(S) + w(S) \ \forall S \subset N$, we must have that for any $\mathbf{x}(v) \in X(v)$, $\mathbf{x}(w) \in X(w)$, $\mathbf{x}(v+w) \in X(v+w)$: $\mathbf{x}(v)_i + \mathbf{x}(w)_i = \mathbf{x}(v+w)_i \ \forall i \in N$.

As it turns out, under the Shapley Restriction, for every $\langle N, v\rangle$, a single unique feasible payoff is assigned to each game:

Theorem. 2.1: The Shapley Value Theorem

Given the coalition game, $\langle N, v\rangle$, the unique feasible payoff that satisfies the Shapley restrictions are:

$$\mathbf{x} = \left(\mathbf{x}_i = \frac{1}{|N|!}\sum_{R\in\mathcal{R}}\left(v(S_i(R)\cup\{i\}) - v(S_i(R))\right), i \in N\right)$$

Where $\mathcal{R}$ is the set of of size $|N|!$ of all permutations of N, and $S_i(R) \subset N$ is the set of all agents $j \in N$ such that they appear before i in the permutation R. This formula is called The Shapley Value.

Proof. We first prove that the Shapley Value satisfies the Shapley Restrictions. In particular, suppose that $v(S\cup\{i\}) = v(S\cup\{j\})\ \forall S \subset N$ for which $i, j \notin S$. Then for each $S_i(R)$, consider $S_j(R')$ where the positions of i and j are interchanged. Then if $j \notin R$, then $i \notin R'$, and we trivially have $v(S_i(R)\cup\{i\}) - v(S_i(R)) = v(S_j(R')\cup\{j\}) - v(S_j(R'))$. If $j \in S_i(R)$, then $S_j(R')$. Write $S_i(R) = S\cup\{j\}$ and $S_j(R) = S\cup\{i\}$. Then we have:

$$v(S_i(R)\cup\{i\}) - v(S_i(R)) = v(S\cup\{j\}\cup\{i\}) - v(S\cup\{j\})$$

and:

$$v(S_j(R)\cup\{j\}) - v(S_i(R)) = v(S\cup\{i\}\cup\{j\}) - v(S\cup\{i\})$$

But since $i, j \notin S$, $v(S\cup\{j\}) = v(S\cup\{i\})$, we must have:

$$v(S_i(R)\cup\{i\}) - v(S_i(R)) = v(S_j(R)\cup\{j\}) - v(S_i(R))$$

Irregardless of if i comes before j vise-versa. What we have done is found for every element in $\mathcal{R}$ when summing for i, an equal element in $\mathcal{R}$ for when summing in j - and so the two sums must be the same.

Next, if $v(S\cup\{i\}) - v(S) = v(\{i\})\ \forall S \subset N, \{i\} \notin S$, then:

$$\mathbf{x}_i = \frac{1}{|N|!}\sum_{R\in\mathcal{R}}\left(v(S_i(R)\cup\{i\}) - v(S_i(R))\right) = \frac{1}{|N|!}\sum_{R\in\mathcal{R}} v\{i\} = v\{i\}$$

And finally, if $u(S) = v(S) + w(S)$, then:

$$\begin{aligned} u(S\cup\{i\}) - u(S) &= v(S\cup\{i\}) - v(S) + w(S\cup\{i\}) - w(S) \\ \Rightarrow \mathbf{x}(v)_i + \mathbf{x}(w)_i &= \mathbf{x}(v+w)_i\ \forall i \in N \end{aligned}$$

Now switching to uniqueness, our plan will be to decompose any game's worth function, v, into a unique "basis", then look at what the Shapley value is under each of these basis elements, then finally use part 3 of the Shapley Restriction (additivity) to show the uniqueness of the Shapley Value. To this end, let $\langle N, v_S\rangle$ be the coalition game defined by $v_S := \chi_S$ - the characteristic function of S. We claim that $\exists!$ $(\alpha_T)_{T\in 2^N}$ such that $v = \sum_{T\in 2^N}\alpha_T v_T$, or equivalently since N finite, that (v_S) linearly independent - I.e. $\sum_{T\in 2^N}\beta_T v_T = 0 \iff \beta_T = 0\ \forall T \in 2^N$. However, if $\beta_T \neq 0$ for some $T \in 2^N$, then $\exists S$ for which $\beta_T = 0\ \forall T \subset S$, but then we have $\sum_{T\in 2^N}\beta_T v_T(S) = \beta_T \neq 0$, a contradiction proving $(\alpha_T)_{T\in 2^N}$ is unique.

Next, we see that under the game $\alpha v_S, \alpha \geq 0$, $\mathbf{x}_i(\alpha v_S) = \alpha/|S|$ if $i \in T$ and 0 otherwise because of the first two sub-restrictions of the Shapley Restrictions. Then since by construction $v = \sum_{T\in 2^N}\alpha_T v_T$, by the third sub-restriction of the Shapley Restrictions, v is determined uniquely. □

3 Infinite Games

3.1 Spaces of Consideration

We now shift our focus to that of infinite game theory. In particular, in the finite case, we assume that the possible coalitions of $N = \{1, \ldots, n\}$ were simply 2^N. Thus, in the infinite setting, we have more flexibility. In particular, let:

Definition. 3.1: General Coalition Game

The measurable space, $(I, \mathcal{C})$, represents a set of players, I, and the set of coalitions - a σ-algebra $\mathcal{C}$ of I. A game on I may be represented by a real-valued set function, $v : \mathcal{C} \to \mathbb{R}$ with $v(\varnothing) = 0$. The usual definitions of monotonic functions and functions of bounded variation are analogous for "games" (If $S \subset T \in \mathcal{C} \Rightarrow v(C) \leq v(T)$, and difference of 2 monotonic games).

Definition. 3.2: Game Spaces

We denote the space of all games of bounded variation as BV. We denote a generic space of games as Q. Q^+ denotes the cone of monotonic members of a space of games Q. A mapping which sends Q into BV is positive if Q^+ gets mapped to BV^+. We denote the subspace of BV of all bounded, finitely additive games as FA. We denote the space of non-atomic measures on $(I, \mathcal{C})$ as NA, and NA^1 if it's also a probability measure. We finally denote the subspace of BV spanned by all powers of NA^+ measures as pNA.

Definition. 3.3: Maps

We denote the group of one-to-one, measurable in both directions, maps of I as G (automorpisms of $(I, \mathcal{C})$). Note that for each $\theta \in G$, θ induces a linear map, denoted θ_* of BV onto itself by $(\theta_* v)(S) = v(\theta S)\ \forall S \in \mathcal{C}$.

Definition. 3.4: Symmetry

A subspace of BV, $Q \subset BV$, is symmetric if $\theta_* Q = Q\ \forall \theta \in G$.

Definition. 3.5: A Value

If Q a symmetric subspace of BV, then a value on Q is a positive linear map, $\psi : Q \to FA$ satisfying:

1. $\psi \theta_* = \theta_* \psi\ \forall \theta \in G$: "Symmetry".
2. $(\psi v)(I) = v(I)\ \forall v \in Q$: "Efficiency".

We also mention that that conditions 1 & 2 are the analogues of conditions 1 & 3 (resp.) in The Shapley Restriction for a finite coalition game.

Values (and in particular it's solution on pNA presented in the next subsection) have historically found use in the operations research literature. To aid in our understanding of these

Example. 3.1: Regulated Monopoly

If we represent I as the different possibilities of a costly good or service to be provided to a consumer (e.g., electricity grids, cell-phone towers, etc). In particular, v would represent the cost to each $S \subset \mathcal{C}$ for configuring the good or service so as to meet S's specifications (e.g., bringing electricity/cell coverage to a small town). Moreover, by imposing the "symmetry" and "efficiency" restrictions above, one can interpret this as a regulation on resource allocation so as to provide the good or service in a "fair" manner (See [2]).

3.2 The Value in pNA

We now present the main result of this section:

Theorem. 3.1: The Aumann-Shapley Value on pNA

There exists a unique value, ψ on pNA. In particular, let μ be a vector of measures in NA and f continuously differentiable on the range, R, of μ, with $f(0) = 0$. Then if $f \circ \mu \in pNA$ and R has full dimension, then:

$$\psi(f \circ \mu)(S) = \sum_{j=1}^{n} \mu_j(S) \int_0^1 (\partial_{x_j} f)(t\mu(I))dt$$

We sketch a proof of establishing the uniqueness, and refer the reader to [1] for the existence and representation proofs.

Proof.
Part 1 of Uniqueness - dense subspace with at most one value: Let $\mu \in NA^1$ and let $G(\mu)$ be the subgroup of automorpisms, G, which are also measure-preserving: $\mu(\theta S) = \mu(S)\ \forall S \in \mathcal{C}$. Then for $u \in FA$ that are fixed points of the action of $G(\mu)$ (i.e., $\theta_* u = u\ \forall \theta \in G(\mu)$) are games of the form $\alpha\mu$, $\alpha \in \mathbb{R}$ (recall $(I, \mathcal{C})$ isomorphic to $([0,1], \mathcal{B}([0,1]))$.

If therefore Q is a symmetric space of games and $\psi : Q \to FA$ is symmetric, $\mu \in NA^1$, and $f : [0,1] \to \mathbb{R}$ is such that $f \circ \mu \in Q$, then $\theta_*(f \circ \mu) = f \circ \mu\ \forall \theta \in G(\mu)$. It follows that the finitely additive game, $\psi(f \circ \mu)$, is a fixed point of $G(\mu)$, which implies that $\psi(f \circ \mu) = \alpha\mu$. If ψ is also efficient, then we also have $\psi(f \circ \mu) = f(1)\mu$. Since the space pNA is a closed linear spans of games of the form $f \circ \mu$, pNA has a dense subspace on which there is at most one value.

Part 2 of Uniqueness - Continuity: To complete this proof, it's sufficient to show that any value on pNA is continuous. For this we need to apply the concept of "internality". We define a linear subspace Q of BV to be <u>reproducing</u> if $Q = Q^+ - Q^+$. It is also said to be <u>internal</u> if for all $v \in Q$:

$$||v|| = \inf \left\{ u(I) + w(I)\ :\ u, q \in Q^+, v = u - w \right\}$$

Trivially, every internal space is reproducing. Also any linear positive mapping ψ from a closed reproducing space Q into BV is continuous, and any linear positive and efficient mapping ψ from an internal space Q into BV has norm 1 ([1] for those details). Therefore, to complete the proof of uniqueness, it is sufficient to show that pNA is indeed internal ([1] for those details). □

In relation to Example 3.1, the unique "fair" allocation for the regulated economy would therefore be given by ψ in Theorem 3.1.

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