Models for Incomplete Observations: Censoring, Truncation and Selection

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Applied Micro - Lecture 12

Incomplete Observations

- Today we study models where the dependent variable is not completely observed
- We study two main cases:
  - censoring: y is censored at some point of the distribution
  - truncation: y is set to missing above some point in the distribution

Censored Data

- A variable can be either top or bottom coded
- Top coded

$$y = \begin{cases} a & \text{if } y^* > a \\ y^* & \text{if } y^* \leq a \end{cases}$$

- Bottom coded

$$y = \begin{cases} b & \text{if } y^* < b \\ y^* & \text{if } y^* \geq b \end{cases}$$

Censored Data - Examples

Censored data can arise for two main reasons.

- First, data artificially top or bottom coded
  - e.g. wages above some level (ceiling on social security contributions)
  - sometimes censoring imposed to prevent identification
- Second, data arise naturally from the problem under consideration
  - e.g. charity donations, people decide not to donate and the distribution shows a mass point at zero
  - in natural censoring, the uncensored variable does not exist, true variable is already censored

Truncated Data

- Similar to censoring, but replaced with missing
- Hence, we have

$$y = \begin{cases} y^* & \text{if } a < y^* < b \\ . & \text{otherwise} \end{cases}$$

- Sometimes truncation due to fact that X are missing

Implications of Censoring in OLS

- Let's consider the model

$$y^* = X\beta + u$$

- Suppose that y^* is the complete variable
- Assume the model satisfies

$$E(u) = 0$$
$$E(X'u) = 0$$

- However, we do not observe y^*

Implications of Censoring in OLS

- The conditional mean or regression function of the OLS is

$$E\left(y^*|X\right) = X\beta$$

- If we run OLS on censored variable we assume that conditional mean is linear
- Consider some censoring

$$y = \begin{cases} y^* & \text{if } y^* > 0 \\ 0 & \text{if } y^* \leq 0 \end{cases}$$

Implications of Censoring in OLS

- The conditional mean can be decomposed as

$$\begin{aligned}E(y|X) &= \Pr(y=0|X)\times 0+\Pr(y>0|X)\,E(y|X,y>0)\\ &= \Pr(y>0|X)\,E(y|X,y>0)\\ &= \Pr(u>-\beta X)\left[X\beta+E(u|u>-X\beta)\right]\end{aligned}$$

- this is not linear!
- We can also rewrite it as

$$E(y|X)=X\beta+\left[\Pr(u>-\beta X)\,E(u|u>-\beta X)-(1-\Pr(u>-\beta X))\,X\beta\right]$$

- Hence, estimation of OLS with censored variable is essentially an OLS with omitted variable!
- Notice that the omitted term is correlated with X

Implications of Truncation in OLS

- Now, consider truncated data

$$y = \begin{cases} y^* & \text{if } y^* > 0 \\ . & \text{if } y^* \leq 0 \end{cases}$$

- Here the conditional mean is

$$\begin{aligned} E(y|X) &= E(y^*|X, y^* > 0) \\ &= E(X\beta + u|X, X\beta + u > 0) \\ &= X\beta + E(u|X, u > -X\beta) \end{aligned}$$

- We have an omitted variable problem

Dealing with Censored Data: Tobit Model

- We now introduce the Tobit model to solve the OLS bias
- As we have seen before when censoring at 0

$$E(y|X) = \Pr(u > -\beta X)\left[X\beta + E(u|u > -X\beta)\right]$$

- Tobit assumptions:
  1. $E(u) = 0$
  2. $E(X'u) = 0$
  3. $u \sim N(0, \sigma^2)$

Dealing with Censored Data: Tobit Model

- The distributional assumption allows to derive the density of $y|X$
- Then we apply maximum likelihood
- The likelihood contribution of censored observations is

$$\Pr(y_i = 0|X_i) = 1 - \Phi(X_i\beta/\sigma)$$

Dealing with Censored Data: Tobit Model

- The likelihood contribution of non-censored observations $(y_i > 0)$ is

$$f\left(y_i | X, y_i > 0\right) = f\left(y_i^* | X, y_i^* > 0\right)$$

- We need to find an expression for f
- Consider the cdf of f

$$F\left(c | y^* > 0\right) = \Pr\left(y^* < c | y^* > 0\right) = \frac{\Pr\left(y^* < c, y^* > 0\right)}{\Pr\left(y^* > 0\right)}$$

$$= \frac{\Pr\left(0 < y^* < c\right)}{\Pr\left(y^* > 0\right)} = \frac{F\left(c\right) - F\left(0\right)}{1 - F\left(0\right)}$$

Dealing with Censored Data: Tobit Model

- f is just the derivative of the cdf

$$f\left(c|X, y^* > 0\right) = \frac{\partial F\left(c|y^* > 0\right)}{\partial c} = \frac{\partial\left[\frac{F(c)-F(0)}{1-F(0)}\right]}{\partial c} = \frac{f\left(c\right)}{1 - F\left(0\right)}$$

- Under the distributional assumptions

$$f\left(c\right) = \frac{1}{\sigma}\phi\left(\frac{c - X\beta}{\sigma}\right) \text{ and } 1 - F\left(0\right) = \Phi\left(\frac{X\beta}{\sigma}\right)$$

Dealing with Censored Data: Tobit Model

- $f(c)$ is the density of a variable that integrates to 1 in $(0, +\infty)$
- We must weight this density for the share of obs above 0
- Hence

$$\begin{aligned}\Pr(y > 0|X) &= \Pr(X\beta + u > 0|X) = \Pr(u > -X\beta|X) \\ &= 1 - \Phi(-X\beta/\sigma) = \Phi(X\beta/\sigma)\end{aligned}$$

- We have

$$\begin{aligned}f(y_i|X_i, y_i > 0) &= \Phi(X_i\beta/\sigma)\, f(y_i|X_i, y_i^* > 0) \\ &= \frac{1}{\sigma}\phi\left(\frac{y_i - X_i\beta}{\sigma}\right)\end{aligned}$$

Tobit Model: Maximum Likelihood

- The individual contribution to the log-likelihood is

$$\ell(\beta, \sigma) = 1(y_i = 0) \ln [1 - \Phi(X_i\beta/\sigma)] + 1(y_i > 0) \ln \left[\frac{1}{\sigma}\phi\left(\frac{y_i - X_i\beta}{\sigma}\right)\right]$$

- The log-likelihood therefore is

$$L(\beta, \sigma) = \sum_{i=1}^{N} \left\{1(y_i = 0) \ln [1 - \Phi(X_i\beta/\sigma)] + 1(y_i > 0) \ln \left[\frac{1}{\sigma}\phi\left(\frac{y_i - X_i\beta}{\sigma}\right)\right]\right\}$$

- The maximization delivers estimates of (β, σ)

Truncated Data Models

- Using a similar procedure, we can write a likelihood function for truncated data
- Let's keep the assumption that $u \sim N\left(0, \sigma^2\right)$
- Take the model truncated below 0

$$y = \begin{cases} y^* & \text{if } y^* > 0 \\ . & \text{otherwise} \end{cases}$$

Truncated Data Models

- We know that the density of the model is

$$f(y|X) = f(y^*|X, y^* > 0) = \frac{f(y)}{1 - F(0)}$$
$$= \frac{\frac{1}{\sigma}\phi\left(\frac{y - X\beta}{\sigma}\right)}{\Phi(X\beta/\sigma)}$$

- The log-likelihood contribution is

$$\ell_i(\beta, \sigma) = -\ln\sigma + \ln\phi\left(\frac{y_i - X_i\beta}{\sigma}\right) - \ln\Phi(X_i\beta/\sigma)$$

- Total log-likelihood is

$$L(\beta, \sigma) = -N\ln\sigma + \sum_{i=1}^{N}\left\{\ln\phi\left(\frac{y_i - X_i\beta}{\sigma}\right) - \ln\Phi(X_i\beta/\sigma)\right\}$$

Comments on Censoring and Truncation

- Censoring is 'better' than truncation
- censored data contain more information about the true underlying distribution
- censored observations are available (i.e. the X's are observable)
- truncated observations are not available

Comments on Censoring and Truncation

- Think about the marginal effects
- The type of marginal effects of main interest depends on the specific analysis
- If interested in effects on y^*, then $E(y^*|X) = X\beta$ and βs are already the marginal effects we need
- If interested in effects on y

$$\text{Censoring: } E(y|X) = \Pr(u > -X\beta)\left[X\beta + E(u|u > -X\beta)\right]$$
$$\text{Truncation: } E(y|X) = X\beta + E(u|u > -X\beta)$$

- When truncation or censoring is "natural" consequence of data structure, we want marginal effect on y
- When it arises because of some artifact, then we probably want marginal effect on y^*

Marginal Effects

- To write the marginal effects, we must write $\mathsf{E}\left(\mathsf{u}|\mathsf{u} > -\mathsf{X}\beta\right)$
- Use the normality assumption on u distribution
- Rule with normal distributions

$$\mathsf{E}\left(\mathsf{z}|\mathsf{z} > \mathsf{c}\right) = \mu + \sigma \frac{\varphi\left(\frac{\mathsf{c}-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{\mathsf{c}-\mu}{\sigma}\right)}$$

- Hence

$$\begin{aligned}\mathsf{E}\left(\mathsf{u}|\mathsf{u} > -\mathsf{X}\beta\right) &= \sigma \frac{\varphi\left(\frac{-\mathsf{X}\beta}{\sigma}\right)}{\Phi\left(\frac{\mathsf{X}\beta}{\sigma}\right)} \\ &= \sigma \cdot \lambda\left(\frac{\mathsf{X}\beta}{\sigma}\right)\end{aligned}$$

- where $\lambda\left(\frac{\mathsf{X}\beta}{\sigma}\right) = \frac{\varphi}{\Phi}$ is called inverse Mills ratio

Marginal Effects

- Using this result, we have

$$\text{Censoring: } \mathsf{E}\left(\mathsf{y}|\mathsf{X}\right) = \Phi\left(\frac{\mathsf{X}\beta}{\sigma}\right)\mathsf{X}\beta + \sigma\varphi\left(\frac{\mathsf{X}\beta}{\sigma}\right)$$

$$\text{Truncation: } \mathsf{E}\left(\mathsf{y}|\mathsf{X}\right) = \mathsf{X}\beta + \sigma\cdot\lambda\left(\frac{\mathsf{X}\beta}{\sigma}\right)$$

- Marginal effects can be easily computed with this formulas

Sample Selection: Heckman Model

- In many cases the sample is not a random draw from the population of interest
- In many applications this is not the case
- Consider the model

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_K x_K + u$$

- where $E(u|X) = 0$

Sample Selection: Heckman Model

- Suppose some info is missing
- we can run the model only on a selected set of N
- Indicator equal to 1 for those observations

$$s_i = \begin{cases} 1 & \text{if } \{y_i, X_i\} \text{ exists} \\ 0 & \text{if } \{y_i, X_i\} \text{ does not exist or is incomplete} \end{cases}$$

Sample Selection: Heckman Model

- Let's write the OLS estimator for this model

$$\hat{\beta}_{OLS} = \left[\sum_{i=1}^{N} s_i X_i' X_i\right]^{-1} \left[\sum_{i=1}^{N} s_i X_i' y_i\right]$$
$$= \beta + \left[\sum_{i=1}^{N} s_i X_i' X_i\right]^{-1} \left[\sum_{i=1}^{N} s_i X_i' u_i\right]$$

- This estimator is consistent only if $E(sX'u) = 0$, which is true if $E(u|s) = 0$
- Hence, u must be independent of the selection process

Random Selection

- Example: suppose that $s \sim \text{Bernoulli}(p)$
- p determines which fraction of the data we select
- you might do this to reduce the computational power needed
- or, data provider might give you only a random sample
- In this case, $E(u|s) = 0$

Deterministic Selection

- Suppose that selection is based on deterministic rule $g(x)$
- e.g. selection is based on age, gender, region, etc.
- Since $E(u|X) = 0$, and s is a function of X, then $E(u|s) = 0$
- Important: Xs that determine selection do not have to be in the dataset

Selection Based on Dependent Variable

- Truncated data arise from sample selection
- Selection based on y
- Hence s is

$$s_i = \begin{cases} 1 & \text{if } a_1 < y < a_2 \\ 0 & \text{otherwise} \end{cases}$$

- Obviously, this selection is not exogenous
- Indeed, $E(u|y)$ cannot be equal to 0 since y is itself a function of u

Endogenous Selection

- Endogenous selection arises whenever $E(u|s) \neq 0$
- e.g. survey data where people asked about income,
- people at the tails of the distribution refuse to answer.
- We only observe income data for those who actually answered the question

Endogenous Selection: Motivating Example

Motivating example in the literature: wages and labor market participation

- Individuals heterogenous in productivity and preference for work
- more productive will receive higher offers
- w_i^0: wage offer received by i
- workers with higher preferences for work have lower reservation wages
- w_i^r: reservation wage for i, lowest w he/she would accept

Endogenous Selection: Motivating Example

- Define w_i^0 and w_i^r as

$$
\begin{aligned}
w_i^0 &= X_{i1}\beta_1 + u_{i1} \\
w_i^r &= X_{i2}\beta_2 + u_{i2}
\end{aligned}
$$

- Assume that $E\left(u_{i1}|X_{i1}\right) = 0$ and $E\left(u_{i2}|X_{i2}\right) = 0$
- We want to estimate β_1, but people work only if wage offer high enough

$$
\begin{aligned}
w_i^0 \geq w_i^r &\Rightarrow \text{i works} \\
w_i^0 < w_i^r &\Rightarrow \text{i is inactive/unemployed}
\end{aligned}
$$

Endogenous Selection: Motivating Example

- In the data we only observe the wage for those who work
- Hence

$$
\begin{aligned}
s_i &= 1\left(w_i^0 \geq w_i^r\right) \\
&= 1\left(X_{i1}\beta_1 + u_{i1} \geq X_{i2}\beta_2 + u_{i2}\right) \\
&= 1\left(Z_i\delta + v_i \geq 0\right)
\end{aligned}
$$

- where $Z_i = (X_{i1}, X_{i2})$, $\delta = (\beta_1, \beta_2)'$ and $v_i = u_{i1} - u_{i2}$
- The model is

$$
\begin{aligned}
w_i^0 &= X_{i1}\beta_1 + u_{i1} \\
s_i &= 1\left(Z_i\delta + v_i \geq 0\right)
\end{aligned}
$$

- Selection is endogenous since v_i depends on u_{i1}

Solving the problem: Heckman Selection

- Let's study a model to solve the selection problem
- This model will only work if we have some data on obs that were not selected
- Take a general model with main equation and selection equation

$$y_i = X_i\beta + u_i$$
$$s_i = 1\left(Z_i\delta + v_i \geq 0\right)$$

- Assume: (s_i, Z_i) always observed for all N
- (y_i, X_i) are observed only if $s_i = 1$
- $E\left(u|X, Z\right) = E\left(v|X, Z\right) = 0$
- $v \sim N\left(0, 1\right)$ (can be relaxed to have $N\left(0, \sigma^2\right)$)
- $E\left(u|v\right) = \gamma v$: imposes a linear structure to conditional mean

Heckman Selection

- Take the conditional mean

$$\begin{aligned} E(y|X, s=1) &= X\beta + E(u|X, s=1) \\ &= X\beta + E(u|X, v > -Z\delta) \end{aligned}$$

- Using the assumptions $u = \gamma v + \xi$, where ξ is non-systematic with zero mean

$$\begin{aligned} E(y|X, s=1) &= X\beta + E(u|X, v > -Z\delta) \\ &= X\beta + E(\gamma v + \xi|X, v > -Z\delta) \\ &= X\beta + \gamma E(v|X, v > -Z\delta) \end{aligned}$$

Heckman Selection

- Now, let's exploit the assumption on v's distribution

$$
\begin{aligned}
E(y|X, s=1) &= X\beta + \gamma E(v|X, v > -Z\delta) \\
&= X\beta + \gamma \frac{\varphi(-Z\delta)}{1 - \Phi(-Z\delta)} \\
&= X\beta + \gamma \frac{\varphi(Z\delta)}{\Phi(Z\delta)} \\
&= X\beta + \gamma \cdot \lambda(Z\delta)
\end{aligned}
$$

- where $\lambda(Z\delta)$ is the inverse Mills ratio
- The true conditional mean includes a second term $\gamma \cdot \lambda(Z\delta)$
- Excluding this term we introduce a bias (X and Z most likely overlap)

Heckman Selection

$$E(y|X, s = 1) = X\beta + \gamma \cdot \lambda(Z\delta)$$

- Heckman: let's include the omitted variable and estimate γ
- However, we must first estimate δ
- Recover the δ from a probit of s_i on Z_i

$$\Pr(s = 1|Z) = \Phi(Z\delta)$$

Heckman Selection

$$\Pr\left(s = 1|Z\right) = \Phi\left(Z\delta\right)$$

- With consistent estimates of δ called $\hat{\delta}$ we have

$$\hat{\lambda}_i = \lambda\left(Z_i\hat{\delta}\right)$$

- Then use it in regression

$$y_i = X_i\beta + \gamma\hat{\lambda}_i + u_i$$

- Standard errors are more complicated since $\hat{\lambda}$ comes from a separate estimate
- Notice: estimating γ you can test endogeneity of selection

Heckman Selection: Additional Comments

- Consider the relationship between X and Z
- May be completely separated or completely identical
- If completely separated omitting $\lambda(Z\delta)$ does not generate OVB
  - OLS on selected sample gives consistent estimates (we still have exogeneity)
  - unless $E[\lambda(Z\delta)] = 0$ the constant will be inconsistent

Heckman Selection: Additional Comments

- If completely identical: $X = Z$
- Problem of multicollinearity: Mills ratio approximately linear

$$E(y|X) \approx X\beta + a + bZ\delta = X(\beta + b\delta) + a$$

- So that cannot estimate β consistently
- Hence, when $X = Z$ identification will only be guaranteed by non-linearity of Mills ratio
- In general, it is better to have $Z = X + Z_1$ so that there are "excluded variables", but all X appear in selection equation
- This is very much like with instrumental variables
- Without Z_1 identification with instrumental variables would be impossible