The Case for Subsidizing Harm: Second-best Pigouvian Taxation with Multiple Externalities

Daniel Jaqua Daniel Schaffa*

November 17, 2019

Abstract

Many activities are subsidized despite generating negative externalities. Examples include needle exchanges and energy production subsidies. We explain this phenomenon by developing a model that generalizes previous work on Pigouvian taxation. In our second-best model the policymaker will optimally subsidize a harmful production activity if a constraint or cost prevents the first-best correction of an even more harmful alternative. We highlight three examples. First, it may be optimal to subsidize a harmful activity if a constraint prevents the taxation of an even more harmful substitute. Second, it may be optimal to subsidize a harmful activity if there is a large administrative cost associated with taxing an even more harmful substitute. Third, it may be optimal to subsidize a harmful production process if the activity mix at lower levels of output uses more harmful activities than the activity mix at higher levels of output. We also show how the functional form of the cost of administering a Pigouvian tax affects the optimal tax.

Keywords: Administrative cost, Corrective tax, Externality, Optimal tax, Optimal tax systems, Pigouvian tax, Second best
JEL Codes: H21, H23

---

*Jaqua: Albion College, danjaqua@gmail.com. Schaffa: the University of Richmond, dschaffa@gmail.com. The authors give special thanks to Jim Hines for his advice and support. The authors thank Steve Bond, Paul Courant, Adam Dearing, Marina Epelman, Morris Hamilton, Louis Kaplow, Tuomas Kosonen, Natalia Lazzati, Ben B. Lockwood, JJ Prescott, Daniel Reck, Nate Seegert, Dan Silverman, Joel Slemrod, Kevin Spiritus, Ugo Troiano, and Mike Zabek for helpful suggestions. The authors also thank the members of the University of Michigan Public Finance community (in particular our seminar attendees), the faculty of the Centre for Business Taxation at Oxford University, and attendees of the 2016 and 2017 NTA Conferences on Taxation, the 2016 IIPF Annual Meeting, the 2017 Mid-Atlantic Junior Faculty Forum, and the 2017 ALEA Annual Meeting for invaluable comments and feedback. Schaffa gratefully acknowledges support from the NIA training grant to the Population Studies Center at the University of Michigan (T32 AG000221). An earlier draft of this paper was titled "Pigouvian Taxation with Costly Administration and Multiple Externalities". Any errors are our own.

Taxes used to correct externality-generating behaviors are named for Arthur Cecil Pigou, who first described many of their features (Pigou, 1920). Pigouvian taxes improve welfare by aligning private incentives to a notion of public wellbeing. A broad class of policies that influence behavior—including carbon taxes, gasoline taxes, and toll roads—fit into a Pigouvian tax framework.

Classical Pigouvian theory directs policymakers to tax activities that generate external harm, setting the tax equal to the marginal harm. Classical Pigouvian theory rejects the idea that complements and substitutes affect the optimal Pigouvian tax. Sandmo (1978), for example, contends that "the fact that a commodity involves a negative externality is not in itself an argument for taxing other commodities which are complementary with it, nor for subsidizing substitutes." Several other papers reaffirm the irrelevance of complementarity, including Kopczuk (2003) and Bovenberg and De Mooij (1994).

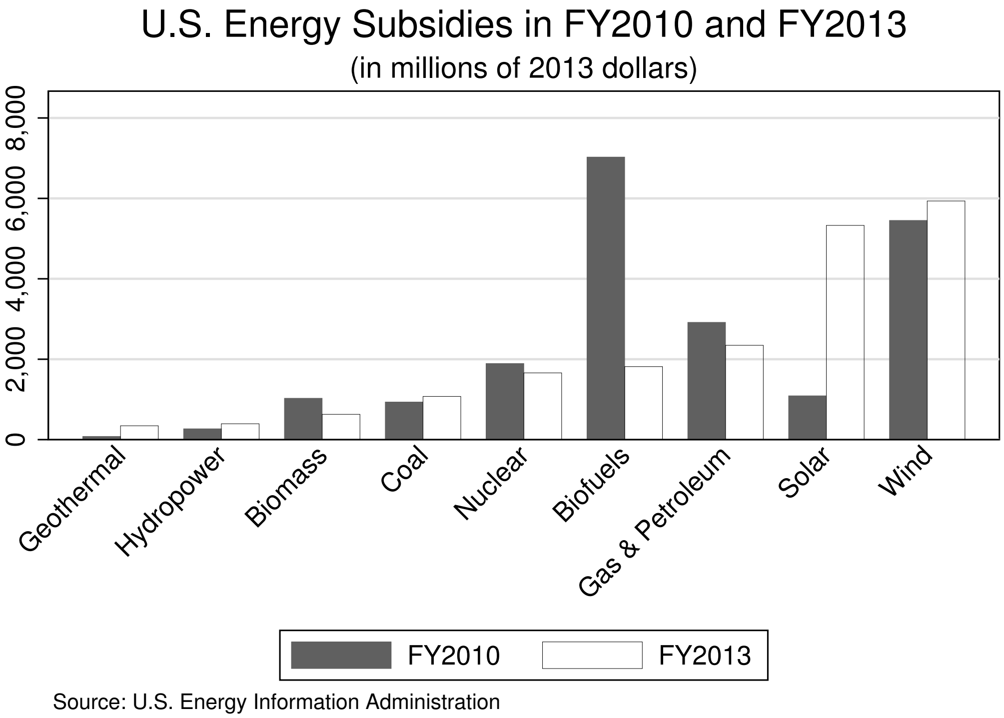
But many harmful activities are subsidized. Important examples include subsidies for needle exchanges and subsidies for most types of energy production despite the fact that all energy production processes generate negative externalities, as shown in the following table.[1]

Table 1: Energy technologies, externalities, and policies

Technology	Externalities	U.S. Policy
Coal	Greenhouse gases, Acid rain, Hazardous waste, Airborne particulates, Risk of mining accidents	Tax & Subsidize
Oil	Greenhouse gases, Hazardous waste, Airborne particulates, Risk of oil spills,	Tax & Subsidize
Natural Gas	Greenhouse gases, Ecosystem destruction, Airborne particulates, Water contamination	Tax & Subsidize
Nuclear	Hazardous waste, Risk of nuclear meltdown	Subsidize
Hydropower	Ecosystem destruction, Risk of dam failure	Subsidize
Bioenergy	Greenhouse gases	Subsidize
Solar	Toxic production process, Ecosystem destruction	Subsidize
Geothermal	Toxic gases released	Subsidize
Wind	Harm to wildlife, Eyesore	Subsidize

[1]This is a partial list of the external harms discussed in the World Energy Council's 2016 World Energy Resource Report.

While this table abstracts from many of the nuances that underlie these policies, it clearly demonstrates that classical (or first-best) Pigouvian analysis is insufficient to explain observed policy. Moreover the subsidies related to energy production exceed $400 billion per year worldwide and substantially lower the private cost of many different energy production technologies as the following graph illustrates.



This paper resolves the apparent discrepancy between theory and practice by placing the Pigouvian taxation of multiple externalities in a second-best setting (Lipsey and Lancaster, 1956).[2] When first-best policy is unavailable, the complementarity between different activities matters. In plausible settings the policymaker will optimally subsidize a harmful activity because a constraint or cost prevents the first-best correction of an even more harmful alternative. By analyzing Pigouvian taxes of multiple activities in the second best, this paper becomes the first to suggest several cases in which an optimal tax system would subsidize externally harmful activities.[3]

Relevant literature. Previous work has found that complementarity matters in other contexts. Corlett and Hague (1953) explain how complementarity matters for incomplete commodity taxation—complementarity with the untaxed good, leisure, affects the optimal revenue raising tax

[2]See Bennear and Stavins (2007) for a description of the many ways that environmental policy is in the second best.

[3]Bovenberg and Goulder (1996) show that a policymaker may optimally subsidize a harmful activity if other taxes are set suboptimally. In our model, the policymaker implements the optimal tax system.

on other commodities. Cremer and Gahvari (1993) describe how the Ramsey equation changes when there is tax evasion. And several papers have shown that there are complementarities between externalities (Burtraw et al., 2003; Groosman, Muller and O'Neill-Toy, 2011; Ambec and Coria, 2018; Fullerton and Karney, 2018).

Most of the research on Pigouvian taxes in the second best has explored how a policymaker should use corrective taxation in the presence of other distortionary taxes (Bovenberg and van der Ploeg, 1994; Bovenberg and Goulder, 1996; Pirttilä and Tuomala, 1997; Cremer, Gahvari and Ladoux, 1998; Goulder, 1998; Parry, 1998; Goulder et al., 1999; Pirttilä, 2000; Cremer and Gahvari, 2001; Gahvari, 2014; Jacobs and de Mooij, 2015). This paper takes as a premise that a corrective tax may not be implementable in its first-best form even in the absence of distortionary taxes.

Some work has explored these inherently second-best Pigouvian taxes.[4] For example, Polinsky and Shavell (1982) consider Pigouvian taxes with one externality and administrative cost. Fullerton and West (2002) study the indirect taxation of a single externality, specifically when emissions are not taxable, but the policymaker can tax gasoline consumption and certain attributes of vehicles. Fullerton and Kinnaman (1995), Fullerton and Mohr (2003), and Fullerton and Wolverton (2005) examine optimal policy when a two-part instrument must be used in lieu of a direct tax on a harmful activity. Holland (2012) shows that when regulation is incomplete an intensity standard may dominate a Pigouvian tax. Jacobsen et al. (Forthcoming) use a sufficient statistics approach that enables comparison between imperfect corrective tax policies.

We extend existing work by modeling multiple externalities in the second best, which allows for the previously unexplored possibility that optimal policy must tradeoff between externalities. Little thought has been given to the optimal Pigouvian taxation of multiple externalities because when the first-best corrective tax is available there is no meaningful distinction between the single and multiple externality cases—each externality can be independently fully corrected.

We also generalize existing work, allowing both direct and indirect taxation and accounting for a wide range of constraints and costs that would push analysis to the second best. In this paper, the policymaker must adhere to constraints while minimizing lost private benefit, external harm, and the administrative cost of taxation. The policymaker must choose the optimal tax system (Slemrod, 1990; Slemrod and Yitzhaki, 2002), selecting both the optimal tax base and the optimal tax rates (Yitzhaki, 1979). As in Mayshar (1991) each implemented tax instrument is used up to the point where its marginal benefit begins to be eclipsed by its marginal harm.

Despite the obvious administrative costs and political constraints that place most policymaking

[4] A parallel literature studies criminal sanctions that are costly to administer and enforce (Polinsky and Shavell, 1992; Kaplow, 1990*a*,*b*).

squarely in the second best, the literature on Pigouvian taxes has not yet explored how multiple externalities should be taxed in the second best. Most externality generating behaviors are complements or substitutes with other externality generating behaviors, and we demonstrate that in the second best the policymaker should take advantage of this complementarity, resulting in novel implications for optimal Pigouvian taxes.

Roadmap. The next section introduces a simple and flexible model with multiple production activities each of which may have an associated externality. Unsurprisingly, the first-best Pigouvian tax on each activity is equal to the marginal external harm from each activity. The remaining sections explore Pigouvian taxation in the second best, emphasizing novel scenarios in which the policymaker would optimally subsidize a harmful activity.

Section two describes the optimal tax when the policymaker faces constraints. The policymaker is first constrained to tax only a subset of production activities, perhaps because she cannot observe all activities or because a powerful lobby protects some activities. If a very harmful activity cannot be taxed, then it may be optimal to subsidize a less harmful substitute. The policymaker is then constrained to tax only output, perhaps because she can collect data on market transactions but not production activities. Even if all production activities are harmful, the policymaker should subsidize output if the activity mix at lower levels of output uses more harmful activities than the activity mix at higher levels of output.[5]

Section three describes how the optimal tax changes when taxes are administratively costly. Administrative costs may (1) be a function of tax rates, (2) be a function of activity levels, (3) have fixed costs, or (4) some combination of (1) - (3). This paper generalizes Polinsky and Shavell (1982), which considers only one activity and therefore cannot explore trade offs between externalities.[6]

If administrative costs are a function of taxes rates, the policymaker should tax every activity. However, the administrative cost imposes a tradeoff on the policymaker. Higher tax rates reduce the externality but increase the administrative cost. In the single activity case, administrative costs that increase with the tax rate always lower the tax relative to the first-best case, which means that the externality is not fully corrected. In the multiple activity case, administrative costs that increase with tax rates lead to taxes which only partially correct the externality. If an activity has a large externality and its tax has a large administrative cost, it may be optimal to subsidize a less harmful substitute.

If administrative costs are a function of activity levels, it may no longer be optimal to tax every

[5]This generalizes Plott (1966) which shows that optimal direct and indirect taxes may have opposite signs if a single harmful activity is used more at lower levels of output.

[6]As Table 2 describes, Polinsky and Shavell (1982) only study a subset of the administrative cost functional forms that we study here.

activity because the reduced external harm may be smaller than the administrative cost and lost private benefit. When it is optimal to tax every activity, the policymaker should set the tax equal to the externality added to the marginal administrative cost. At that tax rate, the private market internalizes both the externality and the administrative cost. When it is not optimal to tax every activity, the policymaker must optimize under incomplete taxation. The analysis of fixed costs follows a similar line of reasoning.

Section four explores how Pigouvian taxation in the second best must be modified to account for a revenue requirement. We show that when the policymaker must contend with externalities and a revenue requirement the optimal tax is a weighted sum of optimal corrective tax expression and the optimal revenue raising tax expression.

1 First-best Pigouvian taxation

This section introduces a model of Pigouvian taxation with multiple externalities and explores optimal policy when the policymaker faces neither constraint nor cost. The result is as expected: the optimal tax vector is equal to the optimal vector of externalities.

Let x be an n-dimensional vector of activities.[7] Activities are used to generate goods and thus indirectly increase utility. Activities also have private costs and can only be performed in nonnegative quantities. Let $X \subseteq \mathbb{R}^n_+$ be the possible set of activities.[8]

Following Ramsey (1927), a net benefit function $b : X \to \mathbb{R}$ maps activity levels to private net benefit. b is twice continuously differentiable, strictly concave, and achieves its maximum somewhere in the interior of X.[9] If x consists of all productive activities, this is a general equilibrium model. Other papers in the literature model private decision making using a utility function subject to a linear constraint. In the appendix, we show that net benefit function maximization generalizes the maximization of a strictly increasing, strictly concave utility function subject to a convex production possibility frontier.[10]

Both external harm and tax burden are linear functions of activities.[11] Let e be the n-dimensional vector of activity externalities and t be the n-dimensional vector of activity taxes. Tax revenues are assumed to be lump-sum redistributed.

[7]We find activity levels the most natural interpretation. However, x could alternatively be interpreted as emission levels, consumption goods, or inputs. One disadvantage of interpreting x as a consumption good is that goods may be consumed in different ways that don't all cause the same external harm (Sandmo, 1978).

[8]This set is convex, open, has finite measure, and lies entirely in the positive orthant.

[9]One way to ensure this outcome is to assume that $\lim_{x_i \searrow \partial X} \frac{\partial b}{\partial x_i} = \infty$ and $\lim_{x_i \nearrow \partial X} \frac{\partial b}{\partial x_i} = -\infty$.

[10]In the second best, the net benefit maximization problem only generalizes the utility maximization problem if there exists an activity, say leisure, that has no externality and cannot be taxed.

[11]Linearity makes the problem more tractable, but the main results are preserved under weak convexity

Proposition 1. *In the first best (when activity taxes are complete and tax administration is costless), the optimal tax vector is equal to the externality vector.*

Proof. The private market[12] solves

$$\max_x b(x) - t^\top x$$

which leads to the first order condition $b'(x) - t^\top = 0$.[13] Because b is strictly concave it has an invertible Hessian. Therefore, by the implicit mapping theorem, there exists a continuously differentiable function, $x(t) : \mathbb{R}^n \to \mathbb{R}^n$, such that $b'(x(t)) - t^\top = 0$. $x(t)$ is the private market's best response function to the tax vector. Note that $b''(x(t))x'(t) = I$, the identity matrix, so $x'(t)$ is also invertible and $x'(t)^{-1} = t'(x)$. The policymaker solves

$$\max_t b(x(t)) - e^\top x(t)$$

which leads to the first order condition $b'(x(t^*))x'(t^*) - e^\top x'(t^*) = 0$. Substituting $b'(x(t^*)) = t^{*\top}$ yields $(t^* - e)^\top x'(t^*) = 0$. $t^* = e$ is clearly a solution, and the invertibility of $x'(t^*)$ ensures that it is the unique solution. □

This is Pigou (1920)'s remarkable result generalized to arbitrary dimensions. When $t = e$, the private market fully internalizes every externality, and the policymaker does not need to know or use information other than the externality vector. Using the tax on activity A to induce changes in activity B is not welfare improving because there is no benefit to changing the activity level in a market that already internalizes the externality.

Proposition 2. *In the first best, $t = e$ is a global maximizer.*[14]

Proof. Consider two arbitrary activity tax vectors, v and w. Assuming no administrative cost, the change in welfare of moving from v to w is

$$\begin{aligned}\Delta b - e^\top \Delta x &= b(x(w)) - b(x(v)) - e^\top [x(w) - x(v)] \\ &= b(x(\gamma(1))) - b(x(\gamma(0))) - e^\top [x(\gamma(1)) - x(\gamma(0))]\end{aligned}$$

[12]We describe the private market as an agent for brevity's sake. More precisely, individuals and firms make choices in the private market that result in an aggregate quantity of activity. We represent their behavior as a maximization problem. The solution to the maximization problem is the market equilibrium quantity of activity.

[13]We use matrix calculus and the associated notation.

[14]This is true because the net benefit function is strictly concave. If activities were perfect complements or substitutes, the optimal tax vector would not be unique. In the second best, even with strictly concave b, it is possible that there exists no finite t^* or that t^* is not unique.

where $\gamma : \mathbb{R} \to \mathbb{R}^n,\ \gamma(r) = v + r(w - v)$. By the fundamental theorem of calculus

$$= \int_0^1 [b'(x(\gamma(r))) - e^\top]\, x'(\gamma(r))\, \gamma'(r)\, dr$$

Recalling that $b'(x(t)) = t^\top$ and $\gamma'(r) = (w - v)$

$$= \int_0^1 (v + r(w - v) - e)^\top\, x'(\gamma(r))\, (w - v)\, dr$$

If $v = e$ (i.e. the policymaker shifts away from the optimal activity tax), then

$$\Delta b - e^\top \Delta x = (w - e)^\top \left[\int_0^1 r\, x'(\gamma(r))\, dr \right] (w - e) \tag{1}$$

The integral is the sum of negative definite matrices and the expression is a quadratic form. Thus the optimal activity tax is strictly superior to any other activity tax. □

Note that the welfare lost increases with the square of the uncorrected externality and increases with the weighted average size of $x'(t)$ over the set $[e, w]$. $x'(t)$ is weighed by r because close to the optimum the marginal benefit and marginal social harm are equal, but they diverge more the further w is from e for the same reason dead weight loss approximations are a triangle.

Proposition 3. *The welfare lost from choosing an arbitrary tax w over the optimal tax can be approximated by $\frac{1}{2}(w - e)^\top \Delta x$.*

Proof. In general, the welfare change of moving from v to w is

$$\Delta b - e^\top \Delta x = \int_0^1 (\gamma(r) - e)^\top\, x'(\gamma(r))\, \gamma'(r)\, dr$$

Integrating by parts

$$\begin{aligned} &= (\gamma(r) - e)^\top\, x(\gamma(r)) \Big|_{r=0}^{1} \; - \; \int_0^1 \gamma'(r)^\top\, x(\gamma(r))\, dr \\ &= (w - e)^\top x(w) - \; (v - e)^\top x(v) \; - \; (w - v)^\top \int_0^1 x(\gamma(r))\, dr \end{aligned}$$

Assuming $\int_0^1 x(\gamma(r))\, dr \approx \frac{1}{2}(x(w) + x(v))$, which is analogous to Harberger (1964)'s approximation, results in

$$
\begin{aligned}
&\approx (w-e)^\top x(w) - \ (v-e)^\top x(v) \ - \ \frac{1}{2}(w-v)^\top (x(w)+x(v)) \\
&= \left[(w-e) - \ \frac{1}{2}(w-v)\right]^\top x(w) - \ \left[(v-e) \ + \ \frac{1}{2}(w-v)\right]^\top x(v) \\
&= \frac{1}{2}(w+v-2e)^\top (x(w)-x(v)) \\
&= \frac{1}{2}(w+v-2e)^\top \Delta x
\end{aligned}
$$

An arbitrary tax, w, is thus approximately $\frac{1}{2}(w-e)^\top \Delta x$ worse than the optimal tax, $v=e$. □

This generalizes the *Harberger Triangle* to a setting with more than one taxed good and an initial set of taxes.[15] Assuming there are no externalities, the approximate change in welfare of moving from v to w is

$$\Delta b \approx \frac{1}{2}(w+v)^\top (x(w)-x(v)) = \frac{1}{2}(w+v)^\top \Delta x \tag{2}$$

This equation corresponds to the sum of *Harberger Trapezoids* associated with a change from an existing set of taxes to another set of taxes. Setting $v=0$ simplifies this equation to the *Harberger Triangle* associated with a change from no taxes to a new set of taxes, in a setting with more than one good and more than one tax

$$\Delta b \approx \frac{1}{2}w^\top (x(w)-x(0)) = \frac{1}{2}w^\top \Delta x \tag{3}$$

2 Constrained Pigouvian taxation

This section modifies the model presented in the previous section by exploring two different constraints. First, the policymaker is constrained to tax only a subset of activities. We call this *incomplete taxation*. Second, the policymaker is constrained to tax only market transactions. We call this *output taxtion* in contrast to *activity taxation*.

A policymaker may not be able to tax some activities because she faces political constraints or because the measurement of some activities is technologically impossible, prohibitively expensive, or particularly susceptible to evasion. Even if no activities can be taxed, the policymaker may be able to tax output, which is easier to measure because of the record keeping associated with market transactions.

[15] See generally Auerbach (1985) and Auerbach and Hines (2002).

2.1 Incomplete taxation

Leaving even one of the relevant activities untaxed alters the analysis because the tax vector no longer sets the marginal social benefit of each activity equal to the marginal social cost of each activity.

Example 1. Let x_s be operating a solar plant and x_c be operating a coal plant. Assume there is only one tax, t_s on x_s. The private market maximizes $b(x_s, x_c) - t_s x_s$, leading to the first order conditions $\frac{\partial b}{\partial x_s} = t_s$ and $\frac{\partial b}{\partial x_c} = 0$. The policymaker's problem is

$$\max_{t_s} b(x_s(t_s), x_c(t_s)) - e_s x_s(t_s) - e_c x_c(t_s)$$

with first order condition $\frac{\partial b}{\partial x_s}\frac{\partial x_s}{\partial t_s} + \frac{\partial b}{\partial x_c}\frac{\partial x_c}{\partial t_s} - e_s\frac{\partial x_s}{\partial t_s} - e_c\frac{\partial x_c}{\partial t_s} = 0$. Substituting the private market first order condition leads to $(t_s^* - e_s)\frac{\partial x_s}{\partial t_s} - e_c\frac{\partial x_c}{\partial t_s} = 0$. Since $\frac{\partial x_s}{\partial t_s} < 0$ the optimal tax is

$$t_s^* = e_s + e_c\frac{\partial x_c}{\partial t_s}\frac{\partial t_s}{\partial x_s}$$

The sign of $\frac{\partial x_c}{\partial t_s}$ is ambiguous and depends on the complementarity of x_s and x_c. If the two activities are substitutes then $\frac{\partial x_c}{\partial t_s} > 0$; if they are complements then $\frac{\partial x_c}{\partial t_s} < 0$. A subsidy will be optimal if the taxed activity, operating the solar plant, has the smaller externality and the two activities are very substitutable. In our example, if the coal plant has a much larger external harm, the solar plant is a substitute, and the policymaker cannot administer a tax on coal, then subsidizing solar energy will be optimal, even though using solar energy to generate electricity has an external harm.

In the above example, the optimal tax depends on complementarity because the marginal private benefit of each untaxed activity will be 0 regardless of that activity's external harm. Because of this uncorrected externality the marginal social benefit of an untaxed activity will not equal that activity's marginal social cost. Using taxes to change the levels of untaxed, externally harmful activities improves welfare. At the optimum, the marginal social benefit of each tax is equal to that tax's marginal social cost. The marginal social benefit of a tax is the reduction in external harm of the taxed activity and all untaxed activities, and the marginal social cost of the tax is the lost private benefit. The following proposition generalizes the example to arbitrary dimensions.

Proposition 4. *If tax administration is costless, the optimal tax on each taxed activity is equal to the externality generated by that activity plus the externalities of all untaxed activities weighted by the responsiveness of the untaxed activity to changes in the taxed activity.*

Proof. Let Θ be the power set of $\{1, ..., n\}$, θ an arbitrary element of Θ, m the dimension of θ, x_θ the m-dimensional vector of taxed activities, $\bar{x}_\theta$ the $n-m$-dimensional vector of untaxed activities,

t_θ the m-dimensional vector of taxes on x_θ, e_θ the m-dimensional vector of externalities generated by x_θ, and $\bar{e}_\theta$ the $n-m$-dimensional vector of externalities generated by $\bar{x}_\theta$. The private market solves

$$\max_x b(x) - t_\theta^\top x_\theta$$

with solution $\frac{\partial b}{\partial x_j} = 0$ for $j \notin \theta$ and $\frac{\partial b}{\partial x_i} = t_i$ for $i \in \theta$. As before, the concavity of b ensures that a continuously differentiable best response function, $x(t_\theta)$, exists. The policymaker solves

$$\max_{t_\theta} b(x(t_\theta)) - e_\theta^\top x_\theta(t_\theta) - \bar{e}_\theta^\top \bar{x}_\theta(t_\theta)$$

which leads to the first order condition $b'(x(t_\theta^*))x'(t_\theta^*) - e_\theta^\top x'_\theta(t_\theta^*) - \bar{e}_\theta^\top \bar{x}'_\theta(t_\theta^*) = 0$. Note that the marginal harm of the tax is equal to the marginal benefit of the tax. Substituting the private market first order condition yields $(t_\theta^* - e_\theta)^\top x'_\theta(t_\theta^*) - \bar{e}_\theta^\top \bar{x}'_\theta(t_\theta^*) = 0$. Rearranging and applying the invertibilty of $x'_\theta(t_\theta^*)$ yields $t_\theta^{*\top} = e_\theta^\top + \bar{e}_\theta^\top \bar{x}'_\theta(t_\theta^*) x'_\theta(t_\theta^*)^{-1}$. In the second best, the optimal tax equations are often implicit equations. For each individual tax we have

$$t_i^* = e_i + \sum_{k\in\theta} \sum_{j\notin\theta} e_j \frac{\partial x_j}{\partial t_k} \frac{\partial t_k}{\partial x_i} \tag{4}$$

□

Consider an example with three activities. Assume that x_1 and x_2 can be taxed, but x_3 cannot be. In that case the optimal tax on x_1 is

$$t_1^* = e_1 + e_3 \frac{\partial x_3}{\partial t_1} \frac{\partial t_1}{\partial x_1} + e_3 \frac{\partial x_3}{\partial t_2} \frac{\partial t_2}{\partial x_1}$$

Note that $\frac{\partial x_2}{\partial t_1}$, $\frac{\partial x_2}{\partial t_2}$, and e_2 do not appear in the expression for t_1^*. The solution of the private market optimization problem implies that t_1 equals the private benefit of a marginal increase in x_1. The harm of a marginal increase in x_1 does include the adjustment of x_3 but does not include the response of x_2 because t_2 is set such that private benefit equals the public harms for a marginal change in x_2.

The special case $m = n$, in which $x'(t_\theta^*) = x'_\theta(t_\theta^*)$, gives $t_\theta^* = e$ as before. If $m < n$ each activity tax may be above or below the associated externality; it may even have the opposite sign as the associated externality.

2.2 Output taxation

An output tax cannot generally induce the private market to select the socially optimal combination of activities. Whereas activity taxes can induce firms to substitute one activity for another, an output tax generally cannot. Consider, for example, producing electricity from either coal or solar energy. Assume using coal is privately cheaper (by an arbitrarily small amount) but also has a larger externality. If coal and solar energy are perfect substitutes, an output tax will not discourage coal use relative to solar use, but an activity tax on coal will.

Output taxes can, however, induce the private market to substitute between activities if different combinations of activities are optimal at different scales of production. A tax on electricity would cause substitution from coal to solar energy if using coal exhibited better economies of scale. Similarly, a subsidy on electricity would cause substitution from coal to solar energy if using solar energy exhibited better economies of scale.

The model used in the previous section is also used in this section albeit with some modification. In the previous section, the model made no explicit reference to output. In fact, the model could implicitly include many different outputs. In this section, we restrict the model to one output and introduce the strictly increasing and weakly concave function $q(x) : X \to \mathbb{R}$ which maps the vector of activities to the quantity of output produced.[16] Let τ be the tax on output. Then the private market's problem is

$$\max_{x} b(x) - \tau q(x)$$

with first order condition $b'(x(\tau)) - \tau q'(x(\tau)) = 0$, where $x(\tau) : \mathbb{R} \to \mathbb{R}^n$ is the private market's best response function to τ.[17] The planner's problem is then

$$\max_{\tau} b(x(\tau)) - e^{\top} x(\tau)$$

with first order condition $b'(x(\tau^*))x'(\tau^*) - e^{\top} x'(\tau^*) = 0$, which sets the marginal benefit of the tax equal to the marginal cost of the tax —where the benefit is reduced external harm and the cost is reduced private net benefit. Substituting in the private market optimum results in

$$\tau^* q'(x(\tau^*))x'(\tau^*) = e^{\top} x'(\tau^*)$$

With a single activity, $x'(\tau^*)$ is a negative scalar. Dividing the planner's problem by $x'(\tau^*)$ and

[16] The appendix contains further discussion of $q(x)$.

[17] The appendix contains a proof that $x(\tau)$ exists and also describes some of $x(\tau)$'s properties.

rearranging yields

$$b'(x(\tau^*)) = e \tag{5}$$

Thus, since at the optimal activity tax $b'(x(t^*)) = e$, the activity tax and output tax can both achieve the first best if there is only one activity. The optimal output tax, $\tau^* = e/q'(x(\tau^*))$, is equal to the marginal externality of output, which is the externality of the activity, divided by the marginal output of the activity at the optimal tax.

When there are multiple activities an output tax cannot generally achieve the first best because $x'(\tau^*)$ will be vector valued. In that case, the optimal tax is a ratio. The numerator is the marginal external harm of an increase in the output tax and may be either positive or negative. The denominator is the marginal output attributable to an increase in the output tax and is always negative.

$$\tau^* = \frac{e^\top x'(\tau^*)}{q'(x(\tau^*))x'(\tau^*)} = \frac{e^\top \mathrm{A} q'(x(\tau^*))^\top}{q'(x(\tau^*))\mathrm{A} q'(x(\tau^*))^\top} \tag{6}$$

where $\mathrm{A} = (b''(x(\tau)) - \tau q''(x(\tau)))^{-1}$ which is negative definite.[18] The output tax will be larger when the activities that decrease the most in response to the output tax have large externalities. The output tax will have smaller magnitude if the activities that are most responsive to the output tax have large marginal products. The output tax will be negative if very harmful activities are relatively less productive at high levels of output.

Example 2. Let x_s be operating a solar plant and x_c be operating a coal plant to produce electricity. Assume there is a tax τ levied on electricity. The private market maximizes $b(x_s, x_c) - \tau q(x_s, x_c)$ with first order conditions $\frac{\partial b}{\partial x_s} = \tau \frac{\partial q}{\partial x_s}$ and $\frac{\partial b}{\partial x_c} = \tau \frac{\partial q}{\partial x_c}$. The policymaker's problem is

$$\max_{\tau} b(x_s(\tau), x_c(\tau)) - e_s x_s(\tau) - e_c x_c(\tau)$$

with first order condition $\frac{\partial b}{\partial x_s}\frac{\partial x_s}{\partial \tau} + \frac{\partial b}{\partial x_c}\frac{\partial x_c}{\partial \tau} - e_s\frac{\partial x_s}{\partial \tau} - e_c\frac{\partial x_c}{\partial \tau} = 0$. Substituting in the private market optimum yeilds $\tau^*\frac{\partial q}{\partial x_s}\frac{\partial x_s}{\partial \tau} + \tau^*\frac{\partial q}{\partial x_c}\frac{\partial x_c}{\partial \tau} - e_s\frac{\partial x_s}{\partial \tau} - e_c\frac{\partial x_c}{\partial \tau} = 0$. Rearranging, the optimal tax is

$$\tau^* = \frac{e_s\frac{\partial x_s}{\partial \tau} + e_c\frac{\partial x_c}{\partial \tau}}{\frac{\partial q}{\partial x_s}\frac{\partial x_s}{\partial \tau} + \frac{\partial q}{\partial x_c}\frac{\partial x_c}{\partial \tau}}$$

By assumption both solar and coal plants pollute ($e_s > 0$ and $e_c > 0$) and more electricity is produced if power plants increase operations ($\frac{\partial q}{\partial x_s} > 0$ and $\frac{\partial q}{\partial x_c} > 0$). The signs of $\frac{\partial x_s}{\partial \tau}$ and $\frac{\partial x_c}{\partial \tau}$ are

[18] Additional details about the output tax problem may be found in the appendix.

ambiguous, but at least one of them must be negative. A subsidy will be optimal if $e_s\frac{\partial x_s}{\partial \tau} + e_c\frac{\partial x_c}{\partial \tau} > 0$, which would happen if, for example, the use of coal increased with the tax $\left(\frac{\partial x_c}{\partial \tau} > 0\right)$, coal produced a large externality relative to solar energy $(e_c > e_s)$, and coal were more responsive to the tax than solar energy $\left(\left|\frac{\partial x_s}{\partial \tau}\right| < \left|\frac{\partial x_c}{\partial \tau}\right|\right)$. Note that this implies that the marginal output of solar energy is larger than coal, $\left(\frac{\partial q}{\partial x_s} > \frac{\partial q}{\partial x_c}\right)$—if not, increasing the output tax would increase output.

In order to make a welfare comparison between the optimal output tax and the optimal activity tax, we find a map from the output tax to activity taxes.

Proposition 5. *For any output tax the private market will respond as if there is a tax on each activity equal to the output tax times the marginal product of that activity.*

Proof. We want to find $t(\tau)$ such that $x(t(\tau)) = x(\tau)$. Recall that $b'(x(t)) = t^\top$ and $b'(x(\tau)) - \tau q'(x(\tau)) = 0$. Thus

$$t(\tau)^\top = \tau q'(x(\tau)) \tag{7}$$

□

Substituting the optimal output tax derived above yields

$$t(\tau^*)^\top = \frac{e^\top x'(\tau^*)}{q'(x(\tau^*))x'(\tau^*)} q'(x(\tau^*)) \tag{8}$$

The higher the marginal product of an activity, the higher the effective tax on that activity. This relationship makes sense because with a tax on output, an increase in an activity causes an increase in the tax burden proportional to the activity's marginal product.

Applying that derivation from the previous section shows that the welfare lost if the policy-maker is constrained to tax only output is

$$\Delta b - e^\top \Delta x = (t(\tau^*) - e)^\top \left[\int_0^1 r\, x'\left(\gamma(r)\right)\, dr\right] (t(\tau^*) - e) \tag{9}$$

where $\gamma(r) = e + r(t(\tau^*) - e)$. Note that the integral is the sum of negative definite matrices and that the expression is a quadratic form. Thus the optimal activity tax is always weakly better than any output tax.[19] Note that the welfare lost increases with the square of the uncorrected externality

[19] If the optimal activity tax is equal to marginal product times the optimal output tax, then both taxes achieve the same welfare because $t(\tau^*) = e$.

and increases with the weighted average size of $x'(t)$ over the set $[e, t(\tau^*)]$. $x'(t)$ is weighed by r for the same reason dead weight loss approximations are a triangle; close to the optimum the marginal benefit and marginal social harm are equal, but they diverge the further $t(\tau^*)$ is from e. Applying the approximation derived in the previous section results in

$$\Delta b - e^\top \Delta x \approx \frac{1}{2}(t(\tau^*) - e)^\top (x(e) - x(\tau^*)) \tag{10}$$

$$= \frac{1}{2}(\tau^* q'(x(\tau^*)) - e)^\top (x(e) - x(\tau^*)) \tag{11}$$

At the optimal activity tax there are no uncorrected externalities. At the optimal output tax the uncorrected externality is $(t(\tau^*) - e)$ or $(\tau^* q'(x(\tau^*)) - e)$. Note that the uncorrected externality is small if activities with large external harms also have high marginal products because the output tax discourages activities with high marginal products relatively more. Using the approximation from the previous section, the average uncorrected externality for each activity is half of the uncorrected externality under the output tax, $\frac{1}{2}(t(\tau^*) - e)$. Multiplying the uncorrected externality by the 'excess' amount of the activity that occurs under the output tax, $(x(e) - x(\tau^*))$ yields the total welfare loss.

The results presented in this section could be generalized further. The planner could be constrained to tax output and a subset of activities. The planner would achieve the first best if she could tax $n - 1$ activities, because the output tax would provide the additional degree of freedom necessary. With fewer activity taxes, the planner would generally remain in the second best. Another possible extension would be to allow multiple outputs, with either complete or incomplete output taxation. In that case, with some functional form assumptions on b and q, the planner would be able to achieve the first best if the number of output taxes equalled or exceeded the number of activities.

3 Costly Pigouvian taxation

Even when policymakers are unconstrained, tax policy will be second-best if corrective taxes carry an administrative cost. This section highlights how the optimal tax is influenced by the functional form of the administrative cost.

Administrative cost includes expenditures on measurement, enforcement, collections, legislation, and litigation.[20] We consider both fixed and variable administrative costs. We call the case

[20] These costs are substantial. Some of these costs appear in the Internal Revenue Service budget, which was over 11 billion dollars for fiscal year 2018.

when administrative cost increases with activity levels *measurement costs*. Measurement costs arise because it is costly to determine the level of pollution generating activities, regardless of who is making these measurements and even if all parties are behaving honestly. Specific examples of these costs include the monitoring devices and the scientists who design and operate them. We call the case when administrative cost increases with tax rates *enforcement costs*. Enforcement costs arise if the incentive to evade increases with tax rates and the government prevents this evasion by pouring more resources into tax enforcement, including more auditors, more lawyers, and more evasion detection software. We call the case when administrative cost increases with tax revenue collected *bureaucracy costs*. Bureaucracy costs resemble the Flypaper effect (Hines and Thaler, 1995)—larger revenues induce larger bureaucracies—for whatever reason the money sticks.

3.1 Fixed administrative costs

If there are fixed administrative costs, it may not be optimal to tax some activities, leading to incomplete taxation. Leaving even one of the relevant activities untaxed alters the analysis (compared to complete taxation) because the tax vector no longer sets the marginal benefit of each activity equal to the marginal harm of each activity—for all untaxed activities, the marginal private benefit is equal to 0. At the optimum each tax must not only account for the activity it is applied to but also every untaxed activity.

If each activity tax has its own fixed cost, the policymaker must optimize the social welfare function 2^n times—once for each possible combination of taxes—using incomplete taxes as described above. Let the fixed cost for each tax be f_i. Recall that the power set of $\{1, ..., n\}$ is Θ. The policymaker's problem is

$$\max_{\theta \in \Theta} \left\{ \max_{t_\theta} b(x(t_\theta)) - e^\top x(t_\theta) - \sum_{i \in \theta} f_i \right\}$$

No closed form solution exists, but the optimal tax expression for each subproblem is

$$t_\theta^{*\top} = e_\theta^\top + \bar{e}_\theta^\top \bar{x}'_\theta(t_\theta^*) x'_\theta(t_\theta^*)^{-1} \tag{12}$$

The planner should choose θ^*, the set of taxes which if set optimally will maximize welfare.

3.2 Variable administrative costs with one activity

In this section, the policymaker must construct an administrative apparatus to collect and enforce taxes. Let c map the activity level and tax rate to administrative cost. Formally c is continuously differentiable and weakly convex and $\arg\min(c) = (0, 0)$. These assumptions are made for tractability, but they are consistent with a planner who employs the most effective tax collecting and enforcement resources first. Administrative cost and marginal administrative cost with respect to x are both increasing in activity level. A subsidy should also be costly to administer, so administrative cost is increasing in the tax rate above 0 and decreasing in the tax rate below 0. However, marginal administrative cost with respect to t is increasing for all t.

The policymaker solves

$$\max_t b(x(t)) - ex(t) - c(x(t), t)$$

which leads to the first order condition $b'(x(t^*))\frac{\partial x}{\partial t} - e\frac{\partial x}{\partial t} - c_1\frac{\partial x}{\partial t} - c_2 = 0$ where c_i denotes the partial derivative of c with respect to its i^{th} argument. Note the general rule: the marginal social benefit of the tax (reduced externality and decreased administrative cost) is equal to the marginal social cost of the tax (reduced private benefit and increased administrative cost). Increasing the tax has an ambiguous effect on administrative cost because increasing the tax increases the enforcement cost but decreases the measurement cost. Substituting $b'(x(t^*)) = t^*$, dividing by $\frac{\partial x}{\partial t}$, and rearranging yields

$$t^* = e + c_1 + c_2\frac{\partial t}{\partial x} \tag{13}$$

The following table presents the optimal tax associated with several possible functions of administrative cost. Note that for all cases with administrative cost the optimal tax equations are implicit equations.

Table 2: Single activity optimal taxes

Case	Cost function	Optimal Pigouvian tax	Previous Reseach
No cost	0	$t^* = e$	Pigou (1920)
Enforcement costs	$c(t)$	$t^* = e + \frac{\partial c}{\partial t}\frac{\partial t}{\partial x}$	
Measurement costs	$c(x)$	$t^* = e + \frac{\partial c}{\partial x}$	Polinsky & Shavell (1982)
Bureaucracy costs	$c(xt) = c(R)$	$t^* = e + \frac{\partial c}{\partial R}\left(t^* + x(t^*)\frac{\partial t}{\partial x}\right)$	Polinsky & Shavell (1982)
Arbitrary costs	$c(x, t)$	$t^* = e + c_1 + c_2\frac{\partial t}{\partial x}$	

When administrative cost takes the $c(t)$ functional form, the marginal administrative cost at $t = 0$ is 0. Thus the policymaker will always implement a tax. However, the optimal tax is not equal to the externality and at the optimum the marginal social benefit of the activity is smaller than the marginal social cost of the activity. This is because when administrative cost is a function of tax rates, higher taxes reduce the external harm but increase administrative cost. The marginal benefit of a higher tax is lower external harm, and the marginal cost of a higher tax is lower private net benefit and higher administrative cost. Thus the tax rate is always lower than the case with no administrative cost,[21] which leaves an uncorrected externality. This shows up in the model because $\frac{\partial c}{\partial t}/\frac{\partial x}{\partial t} < 0$.[22]

When administrative cost takes the $c(x)$ functional form, there is a discrete jump in the administrative cost if policy maker increases the tax from 0—i.e. when the policymaker decides to implement a tax. It is possible that $b(x(0)) - ex(0) > b(x(t^*)) - ex(t^*) - c(x(t^*))$ in which case the optimal tax is $t = 0$, and the externality is uncorrected. When implementing a tax is optimal, the optimal tax is not equal to the externality. If administrative cost is a function of x, the policymaker should raise the tax until the marginal social benefit of the activity equals the marginal social cost of the activity. Because administrative cost increases only with x, the administrative cost can be interpreted as an additional externality. Thus the optimal tax induces the private market to fully internalize the externality and the administrative cost. The tax rate is always higher than the case with no administrative cost.[23] This shows up in the model because $\frac{\partial c}{\partial x} > 0$.

Figures 1-4 depict the optimal activity level and the optimal tax for a single activity. The figures on the left show net private benefit, total externality, and administrative cost in x space. The figures on the right show net private benefit, total externality, and administrative cost in t space. Note that a tax of zero corresponds to the maximum of the private benefit function $b(x)$. Enforcement costs, $c(t)$, are 0 when $t = 0$ and x is at the private optimum. Enforcement costs increase as t gets further away from 0. Measurement costs, $c(x)$, are 0 when $x = 0$ and increase as x increases.

In Figure 1 social welfare, given by $b(x) - e(x)$, is maximized when a tax t^* causes the activity level to decrease to x^*. In Figure 2 social welfare also accounts for $c(t)$, so t^* is lower and x^* is higher than in Figure 1. In contrast, when costs are a function of x, as in Figure 3, t^* is higher and x^* is lower than in Figure 1. Figure 4 displays all of the information on the preceding figures for direct comparison. In Figure 4, subscripts denote which problem that optimum corresponds to.[24]

[21] Similarly, with a positive externality the subsidy rate is always lower than the case with no administrative cost.

[22] For positive externalities $\frac{\partial c}{\partial t}/\frac{\partial x}{\partial t} > 0$. In either case the optimal tax will never be 0 because the marginal administrative cost is 0 at $t = 0$.

[23] However, the optimal subsidy is always smaller than the case with no administrative cost.

[24] fb stands for first best; sbt stands for second best with $c(t)$ administrative costs; and sbx stands for second best with $c(x)$ administrative costs.

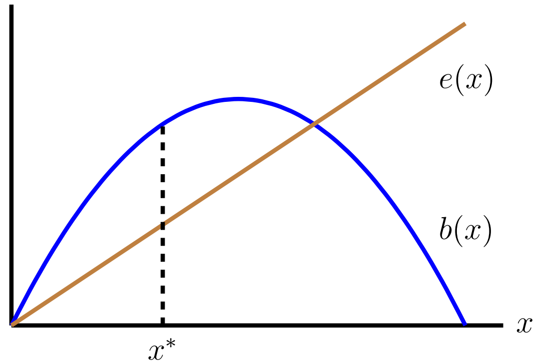


Figure 1.a First best

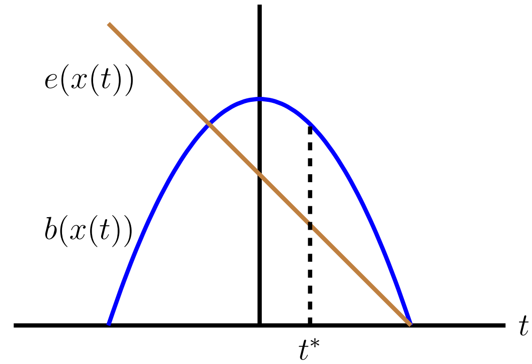


Figure 1.b First best

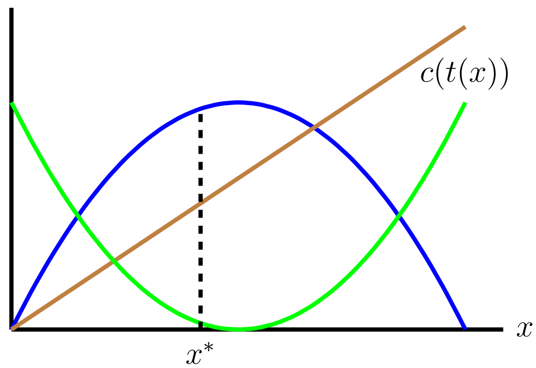


Figure 2.a Enforcement costs

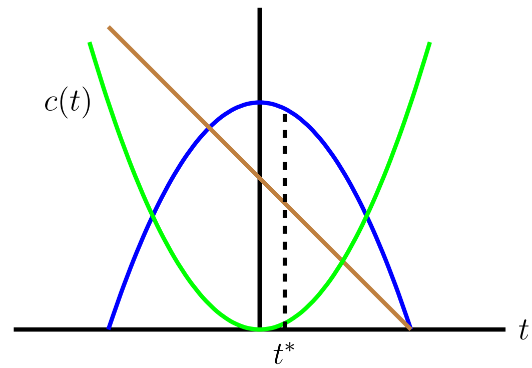


Figure 2.b Enforcement costs

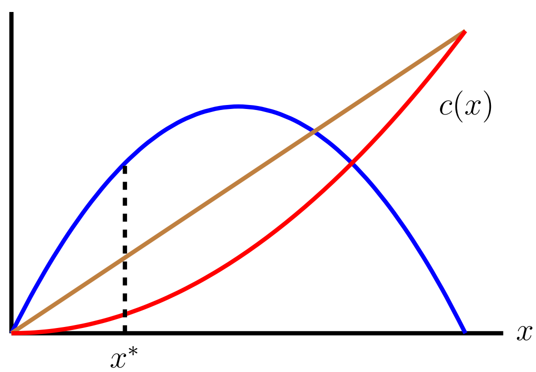


Figure 3.a Measurement costs

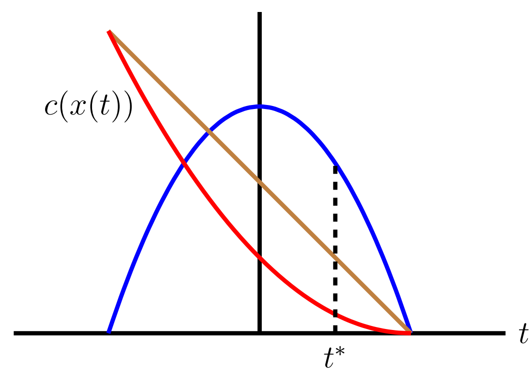


Figure 3.b Measurement costs

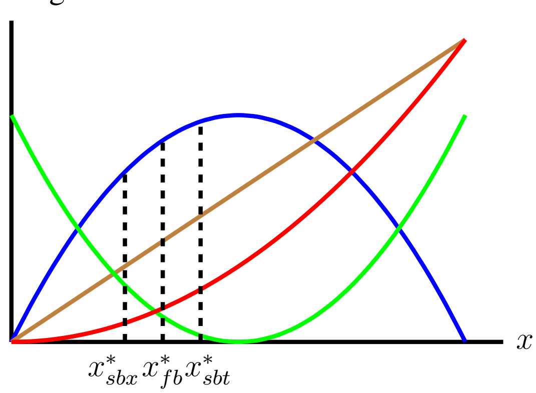


Figure 4.a Comparison

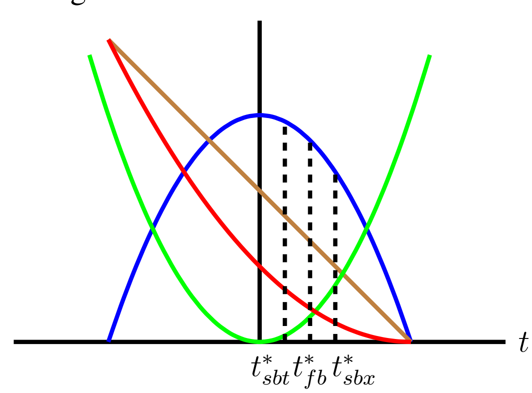


Figure 4.b Comparison

3.3 Variable administrative costs with multiple activities

In this setting optimal taxes depend not only on the externality of the taxed activity and the administrative costs, but also on the complementarity with other activities and the associated externalities.

The cost function $c(x,t)$, $c : X \times \mathbb{R}^n \to \mathbb{R}$, is the multidimensional analogue of the single activity case. We assume that $c(x,t)$ is continuously differentiable, weakly convex, and has $\arg\min(c) = (0,0)$. The private market problem is the same as in the multiple activity, complete taxation, costless administration case. The policymaker solves

$$\max_t b(x(t)) - e^\top x(t) - c(x(t), t)$$

which leads to the first order condition $b'(x(t^*))x'(t^*) - e^\top x'(t^*) - c_1(x(t^*), t^*)x'(t^*) - c_2(x(t^*), t^*) = 0$, where c_1 denotes the partial derivative of c with respect to its first vector argument and c_2 is the partial derivative of c with respect to its second vector argument. Substituting in the private market first order condition and applying $x'(t^*)$'s invertibility yields

$$t^{*\top} = e^\top + c_1 + c_2 x'(t^*)^{-1} \tag{14}$$

The table below describes some special cases. Note that for both cases with administrative cost the optimal tax expressions are implicit formulas.

Table 3: Multiple activity optimal taxes

Case	Cost function	Optimal Pigouvian tax matrix notation	Optimal Pigouvian tax element notation
No administrative cost	$c = 0$	$t^{*\top} = e^\top$	$t_i^* = e_i$
Enforcement costs	$c = c(t)$	$t^{*\top} = e^\top + c'(t^*)x'(t^*)^{-1}$	$t_i^* = e_i + \sum_j \frac{\partial c}{\partial t_j}\frac{\partial t_j}{\partial x_i}$
Measuring costs[25]	$c = c(x)$	$t^{*\top} = e^\top + c'(x(t^*))$	$t_i^* = e_i + \frac{\partial c}{\partial x_i}$

When administrative cost is a function of only tax rates, the optimal tax on each activity is equal to the externality generated by that activity plus the sum of all the marginal administrative costs weighted by the responsiveness of the tax rate to changes in activity. The policymaker should tax each activity because the marginal cost of increasing or decreasing a tax of 0 is 0.[26] However, just as as in the single activity case, an increase in the tax presents a tradeoff: lower externality, but higher administrative cost. Thus, optimal taxes do not induce the private market to fully internalize

[26]It is possible that $t_i^* = e_i + \sum_j \frac{\partial c}{\partial t_j}\frac{\partial t_j}{\partial x_i} = 0$.

the externality. The policymaker should raise (or lower) taxes until the marginal social benefit of the tax is equal to the tax's marginal social harm. In particular, the planner should use taxes with low marginal administrative cost to affect other activity levels. A subsidy may be optimal for some externally harmful activities.[27] c' will take negative values whenever there is a subsidy because the cost of administration will decrease the less negative the tax becomes.

Example 3. Let x_s be operating a solar plant and x_c be operating a coal plant to produce electricity. Assume that both activities are taxed and administrative cost increases with t. The private market maximizes $b(x_s, x_c) - t_s x_s - t_c x_c$, leading to the first order conditions $\frac{\partial b}{\partial x_s} = t_s$ and $\frac{\partial b}{\partial x_c} = t_c$. The concavity of b ensures that $x_s(t_s, t_c)$ and $x_c(t_s, t_c)$ exist. The policymaker's problem is

$$\max_t b(x_s(t_s, t_c), x_c(t_s, t_c)) - e_s x_s(t_s, t_c) - e_c x_c(t_s, t_c) - c(t_s, t_c)$$

with first order conditions

$$\begin{aligned}
&\frac{\partial b}{\partial x_s}\frac{\partial x_s}{\partial t_s} + \frac{\partial b}{\partial x_c}\frac{\partial x_c}{\partial t_s} - e_s\frac{\partial x_s}{\partial t_s} - e_c\frac{\partial x_c}{\partial t_s} - \frac{\partial c}{\partial t_s} = 0 \text{ and} \\
&\frac{\partial b}{\partial x_s}\frac{\partial x_s}{\partial t_c} + \frac{\partial b}{\partial x_c}\frac{\partial x_c}{\partial t_c} - e_1\frac{\partial x_s}{\partial t_c} - e_c\frac{\partial x_c}{\partial t_c} - \frac{\partial c}{\partial t_c} = 0
\end{aligned}$$

Substituting the private market first order condition and manipulating the equation leads to

$$\begin{aligned}
t_s^* &= e_s + \frac{\partial c}{\partial t_s}\frac{\frac{\partial x_c}{\partial t_c}}{\frac{\partial x_s}{\partial t_s}\frac{\partial x_c}{\partial t_c} - \frac{\partial x_s}{\partial t_c}\frac{\partial x_c}{\partial t_s}} - \frac{\partial c}{\partial t_c}\frac{\frac{\partial x_c}{\partial t_s}}{\frac{\partial x_s}{\partial t_s}\frac{\partial x_c}{\partial t_c} - \frac{\partial x_s}{\partial t_c}\frac{\partial x_c}{\partial t_s}} \text{ and} \\
t_c^* &= e_c + \frac{\partial c}{\partial t_c}\frac{\frac{\partial x_s}{\partial t_s}}{\frac{\partial x_s}{\partial t_s}\frac{\partial x_c}{\partial t_c} - \frac{\partial x_s}{\partial t_c}\frac{\partial x_c}{\partial t_s}} - \frac{\partial c}{\partial t_s}\frac{\frac{\partial x_s}{\partial t_c}}{\frac{\partial x_s}{\partial t_s}\frac{\partial x_c}{\partial t_c} - \frac{\partial x_s}{\partial t_s}\frac{\partial x_c}{\partial t_s}}
\end{aligned}$$

Noting that $x'(t)^{-1} = t'(x)$, we have

$$\begin{aligned}
t_s^* &= e_s + \frac{\partial c}{\partial t_s}\frac{\partial t_s}{\partial x_s} + \frac{\partial c}{\partial t_c}\frac{\partial t_c}{\partial x_s} \text{ and} \\
t_c^* &= e_c + \frac{\partial c}{\partial t_c}\frac{\partial t_c}{\partial x_c} + \frac{\partial c}{\partial t_s}\frac{\partial t_s}{\partial x_c}
\end{aligned}$$

A subsidy on solar energy will be optimal if the cost of administering the subsidy is small ($\frac{\partial c}{\partial t_s}$ is negative and has a small magnitude), administering the tax on coal is costly ($\frac{\partial c}{\partial t_c}$ is positive and has a large magnitude), coal is more harmful than solar energy ($e_c > e_s$), and coal x_c and solar energy x_s are very substitutable ($\frac{\partial x_c}{\partial t_s}$ is positive with large magnitude).

[27]The matrix $t'(x(t^*)) = b''(x(t^*))$ describes the effect of a change in the activity vector on tax rates at the optimal tax rate. Because b is concave, this matrix is negative definite, so the diagonals are all negative. An increase in t_i will, thus, reduce x_i although it may increase or have no effect on x_j. This implies that t^* may have negative entries.

When administrative cost is a function of only activity levels, it may not be optimal to tax every activity. This is most apparent if the administrative cost of taxing an activity is avoidable when that activity is untaxed. If the planner decides to tax a particular activity, there will be a discrete increase in the administrative cost. In general, just as in the fixed cost case, the policymaker must optimize 2^n different problems, each with a different variable cost function, c_θ, depending on which activities are taxed. If it is optimal to tax all of the activities, the planner should set the tax equal to the externality plus the marginal administrative cost. At this tax, the private market internalizes both the externality and the administrative tax. If it is not optimal to tax every activity, the complementarity between activities comes into play and it may be optimal to subsidize externally harmful activities.

This section could be generalized to show the optimal tax expression when there are fixed and variable costs and also when there is an output tax with costly administration. The intuition highlighted above remains intact. When there is an output tax and a complete set of activity taxes, the private market's problem does not yield a one-to-one best response function, $x(t,\tau)$—there are infinitely many t's and τ's for which the private market's optimal response is the same x.

4 Revenue requirement

Pigouvian taxes generate revenue in addition to aligning incentives. In this section we show how the optimal taxes described above are altered when there is a revenue constraint. Assume that the revenue requirement is $R \leq t_\theta^\top x_\theta(t_\theta)$. With no externalities, the policymaker's Lagrangian is

$$\mathcal{L} = b(x(t_\theta)) + \lambda(R - t_\theta^\top x_\theta(t_\theta))$$

with first order condition

$$b'(x(t_\theta^*))x'(t_\theta^*) - \lambda(t_\theta^{*\top} x_\theta'(t_\theta^*) + x_\theta(t_\theta^*)^\top) = 0$$

which sets the marginal social cost of each tax equal to the marginal revenue of that tax times the Lagrange multiplier and leads to the optimal tax expression

$$t_\theta^{*\top} = \frac{\lambda}{1-\lambda} x_\theta(t_\theta^*)^\top x_\theta'(t_\theta^*)^{-1} \tag{15}$$

Note that the Karush Kuhn Tucker conditions require that $\lambda \leq 0$, which makes sense since increasing the required revenue should decrease private net benefit. Now we determine the optimal tax when there is a revenue constraint and Pigouvian taxation using the most general model with incomplete taxation and costly administration.

Proposition 6. *The optimal Pigouvian tax with a revenue constraint is equal to the expression for the optimal revenue constrained tax added to the expression for the optimal unconstrained Pigouvian tax times a scaling factor.*

Proof. Starting with the Lagrangian

$$\mathcal{L} = b(x(t_\theta)) - e^\top x(t_\theta) - c(x_\theta(t_\theta), t_\theta) + \lambda(R - t_\theta^\top x_\theta(t_\theta))$$

with first order condition

$$b'(x(t_\theta^*))x'(t_\theta^*) - e^\top x'(t_\theta^*) - c_1 x_\theta'(t_\theta^*) - c_2 - \lambda t_\theta^{*\top} x_\theta'(t_\theta^*) - \lambda x_\theta(t_\theta^*)^\top = 0$$

which sets the marginal social cost of each tax equal to the marginal revenue of that tax times the Lagrange multiplier. Rearranging we have

$$(1-\lambda)t_\theta^{*\top} x_\theta'(t_\theta^*) = e^\top x'(t_\theta^*) + c_1 x_\theta'(t_\theta^*) + c_2 + \lambda x_\theta(t_\theta^*)^\top$$

The optimal tax expression is

$$t_\theta^{*\top} = \frac{1}{1-\lambda} \underbrace{\left(e^\top x'(t_\theta^*) x_\theta'(t_\theta^*)^{-1} + c_1 + c_2 x_\theta'(t_\theta^*)^{-1}\right)}_{\text{optimal Pigouvian tax}} + \underbrace{\frac{\lambda}{1-\lambda} x_\theta(t_\theta^*)^\top x_\theta'(t_\theta^*)^{-1}}_{\text{optimal revenue constrained tax}} \tag{16}$$

□

The Karush Kuhn Tucker conditions require that $\lambda \leq 0$. If the revenue generated by the Pigouvian tax exceeds R, then $\lambda = 0$ and the optimal tax is equal to the Pigouvian tax. The more negative λ is the more the optimal tax policy is dominated by the revenue considerations.

This version of the Pigouvian tax problem has been analyzed by many papers. Sandmo (1975) suggests an additivity property.[28]

> [T]he optimal tax structure is characterized by what might be called an additivity property; the marginal social damage of commodity m enters the tax formula for that com-

[28] See Kopczuk (2003), generalizing Sandmo's additivity property; Bovenberg and De Mooij (1994), describing the relationship between a tax on labor and a Pigouvian tax on a polluting consumption good; and Kaplow (2012), demonstrating that changing a commodity tax to the first-best Piouvian tax while making compensating changes in the income tax creates a Pareto improvement in a model where utility is separable in labor and other activities.

modity additively, and does not enter the tax formulas for the other commodities, regardless of the pattern of complementarity and substitutability.

Our result generalizes Sandmo (1975)'s additivity property. Strictly speaking, Sandmo's additivity may not hold in the second best because complementarity matters in the optimal tax formula.[29] However, the optimal tax is the expression for the optimal Pigouvian tax multiplied by $\frac{1}{1-\lambda}$ added to the expression for the optimal revenue constrained tax.

5 Conclusion

Pigou's seminal insight demonstrated that in a first-best world taxes can fully correct externalities. This paper extends his insight into a second-best world, in which the policymaker faces constraints and costs. In this world there are several cases in which an externality cannot or should not be fully corrected and even cases in which a harmful activity should be subsidized. Intuitively, the policymaker should use the cheapest tax instruments available to her that effect the greatest reduction in external harm.

Our paper also suggests that optimal policy requires more than determining marginal external harm. In the second best, the net private benefit function and administrative cost function are also relevant for policy decisions. While the examples of subsidizing harm suggest that policymakers intuitively understand how to set optimal policy in the second best, our hope is that this paper helps formalize that intuition. Ideally, policymakers will collect the relevant market and administrative cost data to be able to set optimal policy more precisely. Future work could estimate optimal policy parameters and explore optimal specific harm subsidies in greater detail.

References

Ambec, Stefan, and Jessica Coria. 2018. "Policy spillovers in the regulation of multiple pollutants." *Journal of Environmental Economics and Management*, 87: 114–134.

Auerbach, Alan J. 1985. "The theory of excess burden and optimal taxation." *Handbook of Public Economics*, 1: 61–127.

[29] The literature reinforcing complementarity irrelevance is large. See Bovenberg and De Mooij (1994), Bovenberg and van der Ploeg (1994), Bovenberg and Goulder (1996), Fullerton (1997), Pirttilä and Tuomala (1997), Ng (1980), Kopczuk (2003), and Kaplow (2012). A notable exception is Cremer, Gahvari and Ladoux (1998), where complements and substitutes matter in a model that includes heterogeneous consumers and nonlinear external harm.

Auerbach, Alan J, and James R Hines. 2002. "Taxation and economic efficiency." *Handbook of Public Economics*, 3: 1347–1421.

Bennear, Lori Snyder, and Robert N Stavins. 2007. "Second-best theory and the use of multiple policy instruments." *Environmental and Resource Economics*, 37(1): 111–129.

Bovenberg, A Lans, and Lawrence H Goulder. 1996. "Optimal environmental taxation in the presence of other taxes: General-equilibrium analyses." *The American Economic Review*, 985–1000.

Bovenberg, A Lans, and Ruud A De Mooij. 1994. "Environmental levies and distortionary taxation." *The American Economic Review*, 1085–1089.

Bovenberg, Ary Lans, and Frederick van der Ploeg. 1994. "Environmental policy, public finance and the labour market in a second-best world." *Journal of Public Economics*, 55(3): 349–390.

Burtraw, Dallas, Alan Krupnick, Karen Palmer, Anthony Paul, Michael Toman, and Cary Bloyd. 2003. "Ancillary benefits of reduced air pollution in the US from moderate greenhouse gas mitigation policies in the electricity sector." *Journal of Environmental Economics and Management*, 45(3): 650–673.

Corlett, Wilfred J, and Douglas C Hague. 1953. "Complementarity and the excess burden of taxation." *The Review of Economic Studies*, 21–30.

Cremer, Helmuth, and Firouz Gahvari. 1993. "Tax evasion and optimal commodity taxation." *Journal of Public Economics*, 50(2): 261–275.

Cremer, Helmuth, and Firouz Gahvari. 2001. "Second-best taxation of emissions and polluting goods." *Journal of Public Economics*, 80(2): 169–197.

Cremer, Helmuth, Firouz Gahvari, and Norbert Ladoux. 1998. "Externalities and optimal taxation." *Journal of Public Economics*, 70(3): 343–364.

Fullerton, Don. 1997. "Environmental levies and distortionary taxation: Comment." *The American Economic Review*, 245–251.

Fullerton, Don, and Ann Wolverton. 2005. "The two-part instrument in a second-best world." *Journal of Public Economics*, 89(9): 1961–1975.

Fullerton, Don, and Daniel H Karney. 2018. "Multiple pollutants, co-benefits, and suboptimal environmental policies." *Journal of Environmental Economics and Management*, 87: 52–71.

Fullerton, Don, and Robert D Mohr. 2003. "Suggested subsidies are sub-optimal unless combined with an output tax." *Contributions in Economic Analysis & Policy*, 2(1).

Fullerton, Don, and Sarah E West. 2002. "Can taxes on cars and on gasoline mimic an unavailable tax on emissions?" *Journal of Environmental Economics and Management*, 43(1): 135–157.

Fullerton, Don, and Thomas C Kinnaman. 1995. "Garbage, recycling, and illicit burning or dumping." *Journal of environmental economics and management*, 29(1): 78–91.

Gahvari, Firouz. 2014. "Second-best pigouvian taxation: A clarification." *Environmental and Resource Economics*, 59(4): 525–535.

Goulder, Lawrence H. 1998. "Environmental policy making in a second-best setting." *Journal of Applied Economics*, 1(2): 279–328.

Goulder, Lawrence H, Ian WH Parry, Roberton C Williams Iii, and Dallas Burtraw. 1999. "The cost-effectiveness of alternative instruments for environmental protection in a second-best setting." *Journal of public Economics*, 72(3): 329–360.

Groosman, Britt, Nicholas Z Muller, and Erin O'Neill-Toy. 2011. "The ancillary benefits from climate policy in the United States." *Environmental and Resource Economics*, 50(4): 585–603.

Harberger, Arnold C. 1964. "The measurement of waste." *American Economic Review*, 58–76.

Hines, James R, and Richard H Thaler. 1995. "Anomalies: The flypaper effect." *The Journal of Economic Perspectives*, 9(4): 217–226.

Holland, Stephen P. 2012. "Emissions taxes versus intensity standards: Second-best environmental policies with incomplete regulation." *Journal of Environmental Economics and Management*, 63(3): 375–387.

Jacobs, Bas, and Ruud A de Mooij. 2015. "Pigou meets Mirrlees: On the irrelevance of tax distortions for the second-best Pigouvian tax." *Journal of Environmental Economics and Management*, 71: 90–108.

Jacobsen, Mark R., Christopher R. Knittel, James M. Sallee, and Arthur A. van Benthem. Forthcoming. "The use of regression statistics to analyze imperfect pricing policies." *Journal of Political Economy*.

Kaplow, By Louis. 2012. "Optimal control of externalities in the presence of income taxation." *International Economic Review*, 53(2): 487–509.

Kaplow, Louis. 1990*a*. "A note on the optimal use of nonmonetary sanctions." *Journal of Public Economics*, 42(2): 245–247.

Kaplow, Louis. 1990*b*. "Optimal taxation with costly enforcement and evasion." *Journal of Public Economics*, 43(2): 221–236.

Kopczuk, W. 2003. "A Note on optimal taxation in the presence of externalities." *Economics Letters*, 81–86.

Lipsey, R. G., and Kelvin Lancaster. 1956. "The general theory of the second best." *The Review of Economic Studies*, 24(1): 11–32.

Mayshar, Joram. 1991. "Taxation with costly administration." *The Scandinavian Journal of Economics*, 75–88.

Ng, Y. K. 1980. "Optimal corrective taxes or subsidies when revenue raising imposes an excess burden." *The American Economic Review*, 744–751.

Parry, Ian WH. 1998. "A second-best analysis of environmental subsidies." *International Tax and Public Finance*, 5(2): 153–170.

Pigou, Arthur C. 1920. "The economics of welfare, 1920." *McMillan&Co., London*.

Pirttilä, Jukka. 2000. "A many-person corlett-hague tax rule with externalities." *Oxford Economic Papers*, 52(3): 595–605.

Pirttilä, Jukka, and Matti Tuomala. 1997. "Income tax, commodity tax and environmental policy." *International Tax and Public Finance*, 4(3): 379–393.

Plott, Charles R. 1966. "Externalities and corrective taxes." *Economica*, 84–87.

Polinsky, A Mitchell, and Steven Shavell. 1992. "Enforcement costs and the optimal magnitude and probability of fines." *Journal of Law and Economics*, 133–148.

Polinsky, Mitchell, and Steven Shavell. 1982. "Pigouvian taxation with administrative costs." *Journal of Public Economics*, 19(3): 385–394.

Ramsey, Frank P. 1927. "A contribution to the theory of taxation." *The Economic Journal*, 47–61.

Sandmo, Agnar. 1975. "Optimal taxation in the presence of externalities." *Swedish Journal of Economics*, 86–98.

Sandmo, Agnar. 1978. "Direct versus indirect Pigovian taxation." *European Economic Review*, 7(4): 337–349.

Slemrod, Joel. 1990. "Optimal taxation and optimal tax systems." *The Journal of Economic Perspectives*, 4(1): 157–178.

Slemrod, Joel, and Shlomo Yitzhaki. 2002. "Tax avoidance, evasion, and administration." *Handbook of Public Economics*, 3: 1423–1470.

Yitzhaki, Shlomo. 1979. "A note on optimal taxation and administrative costs." *The American Economic Review*, 475–480.

A The net benefit function

The net benefit maximization generalizes the maximization of a strictly increasing, strictly concave utility function subject to a weakly convex production possibility frontier.

Let $u(x, x_{n+1}) : X \in \mathbb{R}^{n+1} \to \mathbb{R}$ be a strictly increasing, strictly concave utility function over the choice set X of a representative agent. Let $p(x, x_{n+1}) : \mathbb{R}^{n+1} \to \mathbb{R}$ be a strictly increasing, weakly convex function, such that $p = 0$ defines a production possibility frontier. Under the implicit mapping theorem, there exists a function f such that for all (x, x_{n+1}) such that $p(x, x_{n+1}) = 0$, $x_{n+1} = f(x)$ and $f_i = -p_i/p_{n+1}$.

Proposition 7. *Define* $b(x) = u(x, f(x))$. *Then*

(i) x^* *maxmizes* b *if and only if* $(x^*, f(x^*))$ *maximizes* u *subject to* $p = 0$*; and*

(ii) $\max_x\{b(x)\} = \max_{x,x_{n+1}}\{u(x, x_{n+1})$ *s.t.* $p(x, x_{n+1}) = 0\}$.

Proof. (i) From the first order conditions of the utility maximization problem $\forall_{i\in\{1,\ldots,n\}} : \frac{u_i}{u_{n+1}} = \frac{p_i}{p_{n+1}}$. The first order conditions of the private net benefit function are $b_i = 0 \implies u_i + u_{n+1}f_i = u_i + u_{n+1}(-p_i/p_{n+1}) = 0 \implies \frac{u_i}{u_{n+1}} = \frac{p_i}{p_{n+1}}$. (ii) follows. □

Proposition 8. $b(x) = u(x, f(x))$ *is strictly concave.*

Proof. First, the weak convexity of p implies the weak concavity of f.

$$p(y, y_{n+1}) - p(x, x_{n+1}) \geq \sum_{i=1}^{n+1} p_i(x, x_{n+1})(y_i - x_i)$$

Thus $\forall (x, x_{n+1}), (y, y_{n+1})$ such that $p(x, x_{n+1}) = p(y, y_{n+1}) = 0$, we have

$$\begin{aligned}
0 &\geq \sum_{i=1}^{n+1} p_i(x, x_{n+1})(y_i - x_i) \implies \\
-p_{n+1}(x, x_{n+1}).(y_{n+1} - x_{n+1}) &\geq \sum_{i=1}^{n} p_i(x, x_{n+1})(y_i - x_i) \implies \\
(y_{n+1} - x_{n+1}) &\leq \sum_{i=1}^{n} (-p_i(x, x_{n+1})/p_{n+1}(x, x_{n+1}))(y_i - x_i) \implies \\
f(y) - f(x) &\leq \sum_{i=1}^{n} f_i(x)(y_i - x_i)
\end{aligned}$$

Second, the weak concavity of f and strict concavity of u imply the strict concavity of b.

$$\begin{aligned}
\forall \theta \in [0,1] : b\left(\theta x + [1-\theta]y\right) &= u(\theta x + [1-\theta]y, f(\theta x + [1-\theta]y)) \\
&\geq u(\theta x + [1-\theta]y, \theta f(x) + [1-\theta]f(y)) \\
&= u(\theta x + [1-\theta]y, \theta x_{n+1} + [1-\theta]y_{n+1}) \\
&> \theta u(x, x_{n+1}) + [1-\theta]u(y, y_{n+1}) \\
&= \theta b(x) + [1-\theta]b(y)
\end{aligned}$$

□

This shows that the unconstrained private net benefit problem generalizes the constrained utility problem. Note that there is no externality from or tax on x_{n+1} in the private net benefit problem. This assumption is innocuous so long as there is at least one activity, such as leisure, that cannot be taxed and causes no externality. Relaxing this assumption makes no difference to the analysis in the first best, where only the relative size of the taxes matter. In that case n tax instruments are sufficient to completely model the benefits and costs of imposing taxes on $n + 1$ activities. In the second best, the absolute size of the taxes may matter, for example when there are administrative costs. We think it safe to assume that leisure is an activity that exhibits no externality and is untaxed.

Using the b function removes prices from the problem, which means that the tax must be in the same units as the objective function.

B The output tax

Net benefit is the benefit of output less the cost of activities. Thus when output is explicitly incorporated into the model, we require two additional functions. Strictly increasing, strictly concave v maps output to private benefit. Strictly increasing, weakly convex g maps the activities to private cost. Thus $b(x) = v(q(x)) - g(x)$.

Proposition 9. *The best response function $x(\tau)$ exists and is continuously differentiable.*

Proof. The private market's problem is

$$\max_x v(q(x)) - g(x) - \tau q(x)$$

with first order condition is $v'(q(x))q'(x) - g'(x) - \tau q'(x) = 0$. Under the implicit mapping theorem, $x(\tau)$ exists and is continuously differentiable everywhere that the derivative of first order condition with respect to x has a non-zero determinant. The derivative of first order condition with respect to x is

$$\begin{aligned} v'(q(x))q''(x) + (q'(x))^\top v''(q(x))q'(x) - g''(x) - \tau q''(x) = \\ [v'(q(x)) - \tau]q''(x) + v''(q(x))(q'(x))^\top q'(x) - g''(x) \end{aligned}$$

$v'(q(x)) - \tau > 0$ under the first order condition. $q''(x)$ is negative semidefinite by assumption. $v''(q(x)) < 0$ by assumption. $(q'(x))^\top q'(x)$ is positive definite because q is strictly increasing. And $g''(x)$ is positive semidefinite by assumption. Thus

$$= \underbrace{[v'(q(x)) - \tau]}_{+} \underbrace{q''(x)}_{\text{NSD}} + \underbrace{v''(q(x))}_{-} \underbrace{(q'(x))^\top q'(x)}_{\text{PD}} - \underbrace{g''(x)}_{\text{PSD}}$$

Because a positive (semi)definite matrix multiplied by a negative scalar is a negative (semi)definite matrix and since the sum of a negative definite matrix and a negative semidefinite matrix is a negative definite matrix, the sum here is negative definite. Since a negative definite matrix has a non-zero determinant, $x(\tau)$ exists and is continuously differentiable.

□

Proposition 10. $q'(x(\tau))x'(\tau) < 0$

Proof. Taking the derivative of the private market's first order condition with respect to τ yields

$$\begin{aligned} v'q''x' + (q')^\top v''q'x' - g''x' - \tau q''x' - (q')^\top = 0 \implies \\ (x')^\top v'q''x' + (x')^\top (q')^\top v''q'x' - (x')^\top g''x' - (x')^\top \tau q''x' = (q')^\top \implies \\ \underbrace{(v' - \tau)}_{+} \underbrace{(x')^\top q''x'}_{-\text{ or }0} + \underbrace{(q'x')^2}_{+} \underbrace{v''}_{-} - \underbrace{(x')^\top g''x'}_{+\text{ or }0} = (q'x')^\top \implies \\ (q'x')^\top < 0 \end{aligned}$$

□