ECARDA

Building Numeracy

Some suggestions for teaching basic arithmetic

This paper brings together the ideas and materials used in a series of working sessions with teachers from primary schools across the country during the period 1996 - 1998. First published as Book 4 in a series called "Putting Learning First", the paper became popular in initial and continuous teacher training.

Peter Lacey
September 1998

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ACKNOWLEDGEMENTS

This book is written in response to three chains of events with which I have recently been closely involved.

1. An analysis of what constitutes successful practice in the teaching of numeracy in schools in North East Lincolnshire Local Education Authority.
2. Observations of teachers teaching.
3. Working with colleagues in the Association of Teachers of Mathematics.

I fully acknowledge all these sources, though what I have learnt and written in this book is a personal construction. I do not write here in a professional capacity, nor am I representing any particular professional association.

In 1997, I designed a questionnaire, which was used by the advisory service to survey the teaching of numeracy in all the schools in North East Lincolnshire. The checklist of fifteen items used in the questionnaire is included at the back of this book for your reference. (See Appendix 2) As well as writing a commentary, advisers made a judgement on a one-to-five scale, related to the observed incidence of each of the fifteen items.

A synoptic report on the teaching of numeracy across the LEA has acted as a basis for discussion. In addition, our LEA statistician, Frank Ford, took the quantitative judgements and measured which set of items were most closely associated with high performance in national tests and examinations. For eleven year olds, the following features appear to be the keys to success:

- An emphasis on teaching mental methods of calculating (not just testing);
- An emphasis on discussing and explaining methods of calculating;
- Pupils work with resources which allow them to visualise the structure of number;
- The sequence of teaching, set out in the scheme of work, is well defined.

This book is a response to these findings: an attempt to disseminate not only good practice, but successful practice — practice that produces results.

I acknowledge the effort of countless teachers whom I have seen at work. Watching teachers apply their craft is a privilege. I have learnt much from discussing the teaching and learning of mathematics with teachers and pupils.

I acknowledge the insights I have gained from discussing the teaching and learning of mathematics with colleagues, particularly those in the Association of Teachers of Mathematics (ATM). A conference on the teaching of arithmetic, led by Geoff Faux in 1997, was an opportunity for many of us to refine and articulate our thinking in preparation for the Government's numeracy initiative. I was one of the speakers at the conference, but I acknowledge the contributions of the other speakers, particularly Geoff Faux, Marjorie Gorman, Ian Harris, Dave Hewitt and Dick Tahta.

Thanks are also due to HMSO for permission to use a number of extracts from their publications

1. INTRODUCTION

For over a hundred years, criticism has been levelled at pupils' abilities in working with number. Just look at some of the quotes below:

EVEN THE 'GOOD' TIMES WERE BAD!!

A. "I must confess to some surprise at the extremely poor result in arithmetic."
(H.M.I. Report: Kent and Sussex 1875)

B. "In arithmetic, I regret to say worse results than ever before have been obtained …."
(H.M.I. Report: Stafford and Derby 1876)

C. " far too much time is given to the mechanical part of the subject. The result of this unintelligent teaching shows itself in the inability of the upper standards to solve very simple problems."
(H.M.I. Report 1895)

D. "… accuracy in the manipulation of figures does not reach the same standard which was reached twenty years ago. Some employers express surprise and concern at the inability of young persons to solve simple numerical operations involved in business."
(Board of Education Report 1925)

E. "… Experience shows that a large proportion of entrants have forgotten how to deal with simple vulgar and decimal fractions, have very hazy ideas on some arithmetical processes, and retain no trace of knowledge of algebra, graphs or geometry, if, in fact, they ever did possess any. Some improvements in this position may be expected as a result of raising the school leaving age, but there is as yet no evidence of any marked change."
(Mathematics Association Report 1954)

F. "Undoubtedly, mathematics in this country is in a desperate plight;"
(W.A.Gibby 1962)

G. "It is therefore clear that criticism of mathematical education is not new."
(Cockcroft Report 1982)

It appears that there has never been a 'golden age' in teaching mathematics.

Recent evidence shows that when comparing our number abilities with those of other countries in the world, questions of concern continue to be raised. In seeking improvement, clearly, a whole-scale return to using past teaching methods is not going to be productive. And, importing packages of methodology from abroad, fails to take into account the social and cultural setting of our own education system. We need a radical overhaul of the teaching of mathematics, selecting and adapting those ingredients which have proved to be successful.

This book unashamedly focuses on teaching. I have become alarmed by the proposition that all that is needed is a set of smart activities. I have seen teachers attend courses and been more than satisfied when given a set of activities they can use in the classroom. I have observed lessons where pupils have been completely engaged in working on these activities. The key question is how can these engaging activities be used to promote a progression of learning. I would argue that engaging activities are a necessary but not sufficient condition to promote learning. Sufficiency is provided through using these activities as teaching aids; by helping teachers identify where pupils are and where they need to go next. Sufficiency is provided through teaching.

This book is not an academic thesis, but a collection of successful teaching approaches I have noticed when observing lessons. I am not proposing anything new, but offering a common sense synthesis of methods which draws on both recent and past experience. Teachers have been tempted onto "bandwagons" over the duration of my career in education and I believe it is high time that a stand is made for common sense. If a teaching approach works, in that pupils become more confident and competent mathematicians, then that method deserves consideration.

This book intends to help teachers gain a better grasp of what mathematics is really about, and no compromise has been made with the integrity of the subject. Understanding of the subject empowers teachers to use the skills they have in teaching. If 'smart' methods for, say, subtraction are proposed, then that smartness is explained and linked to the perfect regularity of the number system from which it came.

In explaining what I mean by the essence of the subject and its integrity, I have chosen to list the big ideas in mathematics which have been proposed and published by Geoff Faux in the Association of Teachers of Mathematics' journal "Mathematics Teaching" in 1998.

BIG IDEAS IN MATHEMATICS

Pupils should appreciate:

- that the number system is perfectly regular;
- that mathematics is shot through with infinity;
- . that a lot can be gained from a little;
- equivalence;
- inverse

The bulk of this book focuses on the teaching of number skills; the basic toolkit for survival in today's world. But, as important, is pupils' ability to use that tool kit, not only to describe situations and solve problems outside the mathematics lesson, but also to explore further the world of mathematics itself.

Many of the teaching approaches proposed in this book are more efficiently translated into practice through working with a whole class, or with large groups of pupils. Demonstration and explanation of ideas are often coupled with whole class responses, discussion and practice.

This is not a text book but it is a book about teaching mathematics.

2. ABOUT NUMERACY

Various definitions. The primacy of confidence. A working description. The balance between acquisition and application.

The word 'numeracy' was first used officially in the Crowther Report in 1959. Since that time it has been variously described. (see below)

WHAT IS NUMERACY?

1. "…A high degree of ability to cope with current mathematical demands on the community."
(*Crowther 1959*)

2. "…an 'at-homeness' with numbers and an ability to cope with the mathematical demands of everyday life …An ability to have some appreciation and understanding of information which is presented in mathematical terms, for instance, graphs, charts or tables, or by reference to percentage increase or decrease."
(*Cockcroft Report 1982*)

3. "…includes the ability to:
   - understand and interpret numerical data which is presented in a range of forms;
   - present numerical data in a range of forms appropriate to different purposes and audiences;
   - select and apply numerical methods to problem solving."

   (*NCC Core Skills 16 – 19 1990*)

4. "…an ability to deal comfortably with the fundamental notions of number and chance."
(*Paulos in "Innumeracy" 1990*)

5. "…the pupils' ability to cope with the mathematical demands of everyday life and to:
   - handle number and measurement fluently, mentally, orally and in writing;
   - use calculators accurately and appropriately;
   - apply spatial concepts where necessary;
   - make sense of information presented numerically and graphically;
   - handle statistical information in everyday contexts."

   (*Ofsted Inspection Handbook 1993*)

I have started countless training sessions for teachers, governors and parents with a short test. I have suggested that I need to know what they know and can do so that the session can address their needs. It has always been a controversial start, with the audience becoming increasingly agitated. I have always cut short the test before the end of the ten questions and asked the audience how they felt. Rarely have the whole audience seen this as a positive experience. Indeed, many have talked about feelings of fear and inadequacy.

ATTITUDES TO MATHEMATICS

"The extent to which the need to undertake even an apparently simple and straightforward piece of mathematics could induce feelings of anxiety, helplessness, fear and even guilt in some of those interviewed was, perhaps, the most striking feature of the study."
(Cockcroft Report 1982, quoting the findings of Bridgid Sewell's survey on The Use of Mathematics by Adults in Daily Life.)

"Once attitudes have been formed, they can be very persistent and difficult to change. Positive attitudes assist the learning of mathematics; negative attitudes not only inhibit learning but …… very often persist into adult life …."
(Cockcroft Report 1982)

Clearly, one of the elements of being numerate is connected to how we feel about number. John Paulos, writing about innumeracy, observed that the word 'number' was often interpreted as 'numb-er'; that which numbs! Confidence with number must be one of the key ingredients of numeracy. Other ingredients of numeracy are listed below:

INGREDIENTS OF NUMERACY

CONFIDENCE

COMPETENCE

APPLICATION

MATHEMATICAL SKILLS

EVERYDAY SITUATIONS

We can combine these ingredients to formulate a working definition of numeracy.

DEFINITION OF NUMERACY

NUMERACY

is a

CONFIDENCE

and

COMPETENCE

in the

APPLICATION

of

MATHEMATICAL SKILLS

in

EVERYDAY SITUATIONS

A definition of numeracy is helpful when devising a whole-school numeracy policy. It clearly indicates that the whole curriculum has a part to play in developing numeracy, and this needs to be planned for. Equally clear, is the signal to what should be going on in mathematics lessons, so that pupils are properly prepared to deal with situations which occur in other lessons as well with situations which occur beyond school. By listing some of the features of numeracy, the priorities for mathematics teaching, and arithmetic teaching in particular, are made evident.

FEATURES OF NUMERACY

- An at-homeness with number; knowing how numbers relate to each other.
- Visualising number lines or squares or other patterns; being able to 'see' in your head.
- Using mathematical language to communicate ideas orally, and then in writing, using words, diagrams and graphs.
- Using simple mathematics to describe situations and to solve problems outside and inside school.
- Mentally calculating, and developing written methods for more complex calculations.

There is a relationship between the acquisition of mathematical skills and the application of mathematical skills. Numeracy is a blend between the 'can do' and 'can use'. The National Curriculum programmes of study set out the 'can do' elements in sections on Number, Geometry, Algebra and Handling Data. It attempts to spell out the 'can use' elements in the section on Using and Applying Mathematics.

Being numerate implies being able to use a toolkit of mathematical knowledge, skills and understanding to solve problems.

Numeracy (like literacy and IT competence) is a key skill in that it unlocks doors to further learning.

Think of the 'can do' elements as forming a tool-kit. A tool kit is only useful to the extent that the tools can be used, with confidence, to solve problems. I may be better off being able to use a simple tool kit creatively, confidently and in a range of different situations than to have an advanced tool kit and lack the confidence to use it.

My family, bless them, continue to buy me sophisticated electrically driven tools with the hope that I will improve my DIY skills. I must admit that some remain in the garage in their original wrappers, still untouched! And I also must admit that I have an over dependence on the hammer. I have been known to use one to insert reluctant screws! But, I can get those essential jobs done with hammer, screwdrivers and pliers!!!

Focusing only on increasing the number of tools does not make a person numerate.

I may be better off being able to use a modest mathematical toolkit confidently, creatively and effectively in a wide range of different situations, rather than to have a large toolkit and lack the confidence or ability to use it.

The point I am making is serious. Are pupils better off being taught to use a modest tool kit with confidence in a wide range of situations, or being taught an increasing number of mathematical skills which they are unable to use? Progression in numeracy is, in one part, to do with using a given set of tools in an increasing range and complexity of situations, and, in another part, is to do with building up the tool kit itself. The whole curriculum can help pupils learn to use their tool kit of mathematical skills.

The rest of this book focuses on developing the tool kit in such a way that pupils gain the skills and confidence to use them.

3. ABOUT MENTAL ARITHMETIC

The essential tool-kit. Recall and agility. Seeing images of the number structure in the head. Using these images to manipulate numbers. Knowing how a process works is more efficient than memorising all possible outcomes of that process – an activity for teachers.

I want to focus on the teaching of calculating skills. I use the word 'teaching' on purpose. As I have mentioned earlier, I think there is a sufficient supply of activities in the system, and previous numeracy initiatives have focused on providing rich, useful and helpful activities for use in the classroom. These are necessary but not sufficient for bringing about improvement. Learning is secured by incorporating these activities into teaching strategies.

I want to focus further down onto the development of mental arithmetic skills for these should underpin the development of written methods (and not be at odds with them). Mental arithmetic skills are the essential skills in the basic tool kit. And before I go further, this is what I mean by mental arithmetic:

MENTAL ARITHMETIC

1. MENTAL RECALL

Early understandings and constructions of mathematics become embedded as facts.

Pupils should be able to recall these facts in order to solve efficiently more complex calculations.

e.g. 17 x 8
by recall of (10 x 8) and (7 x 8)
and then find their sum.

15 ÷ 0.3
by recall of 15 ÷ 3
and then work out the place value of the quotient.

2. MENTAL AGILITY

Made up of: VISUALISATION
STRATEGY

In coming to understand the relationship between numbers, pupils may visualise, say, a number line or number square, and steps of equal amount. They develop appropriate strategies, which relate to how they 'see' the problem.

e.g. In working out 63 – 38, they may see 38 as 2 less than 40.
They go on to subtract 40 and then add 2.

In working out 7003 – 6940, they may see these numbers on a number line.
They go on to add 60 to 3.

In many other countries, they do not teach written methods until children are 8 years old. All work until then is undertaken mentally. That does not mean they do not write anything down. They do. They write down mathematical statements which describe situations and they write to communicate what they have done. 3 + 5 = 8 is not a written calculation, it is a statement of a calculation undertaken mentally.

Recall of number facts does not necessary imply teaching only memorising. From my own experience, when trying to recall someone's name, I often find myself trying to bring to mind the context in which I last encountered the person. If I can recall the location, or other people in the group, then recalling the person's name is made easier. In a similar way, knowing the location of numbers in a particular image of the number structure facilitates recalling certain number facts.

I find calculating mentally with numbers is also made easier by being able to bring to mind particular images of the number structure. Some may argue that this understanding is about the algebra of arithmetic. I do not disagree with this proposition. However, in 'seeing' this structure, I find it helpful to use and teach visible manifestations of the structure. They have a shape and form and, in this sense I would claim that there is a geometry in arithmetic.

Because the structure of number is perfectly regular, it is possible to mentally undertake a calculation never before encountered. For example, in a recent mental test, pupils were asked to mentally calculate 7.3 x 100. They were given five seconds. How did they do it? Not by trying to recall the last time they multiplied 7.3 x 100, but by applying an understanding of the effect of multiplying a number by a power of ten. At some point, this understanding will have been achieved through teaching. The key question is: "What sort of teaching can bring about confident understanding of number?"

This book is about teaching images of the number structure so that it is known and understood by pupils. In this way pupils are empowered and enabled to work with numbers in their head.

An activity for teachers

In order to illustrate this point, you are invited to engage in a short experiment. For part one of the experiment, look at the information in the box below for 90 seconds. You may not write anything down. Do not turn over the page as that will spoil the experiment.

Look at these operation tables for 90 seconds.
You are going to be given a test.

2*1 = 5	3*1 = 11	4*1 = 19	5*1 = 29	6*1 = 41
2*2 = 7	3*2 = 14	4*2 = 23	5*2 = 34	6*2 = 47
2*3 = 9	3*3 = 17	4*3 = 27	5*3 = 39	6*3 = 53
2*4 = 11	3*4 = 20	4*4 = 31	5*4 = 44	6*4 = 59
2*5 = 13	3*5 = 23	4*5 = 35	5*5 = 49	6*5 = 65
2*6 = 15	3*6 = 26	4*6 = 39	5*6 = 54	6*6 = 71
2*7 = 17	3*7 = 29	4*7 = 43	5*7 = 59	6*7 = 77
2*8 = 19	3*8 = 32	4*8 = 47	5*8 = 64	6*8 = 83
2*9 = 21	3*9 = 35	4*9 = 51	5*9 = 69	6*9 = 89

Now turn over the page, and, without referring to this table, answer the following ten questions, you may give yourself up to 10 seconds for each question.

1.	4 * 6
2.	3 * 7
3.	5 * 2
4.	6 * 5
5.	5 * 9
6.	2 * 6
7.	5 * 4
8.	4 * 5
9.	3 * 3
10.	5 * 8

Now check your answers from the table, and give yourself a mark out of ten!

For part two of the experiment, look at the information in the box below for 90 seconds. Do not read any other information.

Look at the operation described below for 90 seconds.
You are going to be given a test.

* is defined such that a*b = (a+b) x a - 1

in other words: add the two numbers together, multiply by the first number, and then subtract 1.
E.g. 3*7

3 add 7 is 10, 10 times 3 is 30, now subtract 1 to get 29
so, 3 * 7 = 29

Now work out 6*3.

6 add 3 is 9, 9 times 6 is 54, now subtract 1
so, 6 * 3 = 53

Now cover the box above, and answer the ten questions above. Check your answers from the table and give yourself a mark out of ten. What do you notice?

I have used this activity with teachers, when I give half the audience the information in the first box, and the other half of the audience the information in the second box. I present the test orally to the whole audience and then read out the answers. On the whole, the teachers who are given the information in the second box score more highly.

This is interesting in that there are fewer pieces of information in the second box, and yet, this smaller quantity of information is more empowering than all the information in the first box. Remember the big ideas in mathematics, listed at the beginning of this book, that a lot can be gained from a little.

Caleb Gattegno, whose name is associated with the tens chart above, spent his life researching how we learn. He was particularly interested in the economy of learning. He argued that every fact that has to be memorised has a “cost” in terms of effort. He even invented a unit of cost, which he called the *Ogden*, named after one of his students! To secure learning he advocated finding the most economic methods.

The experiment you have just undertaken illustrates that knowing how a process works is more economic than memorising a set of possible outcomes of that process. In terms of number, knowing how the number system is structured is likely to be more productive than only memorising some given outcomes. Of course, in reality, we will memorise those outcomes which we use more frequently. What I am saying is that memory comes from frequent use and is not a replacement for teaching the structure of the system.

Calculating is employing a method to short cut counting. Any calculation can be reconstituted into a count. Think of 3 x 8. This could be undertaken by counting 3 rows of 8 counters. But it all gets a bit tedious when I am counting 29 x 34!!

Or think of 258 + 397 using counters.

4. EARLY COUNTING

Learning number names. Using number names for ordering. Using number names to describe a quantity.

If calculating is short cut counting, then the ability to count is a prerequisite to the ability to calculate. But counting is pretty challenging. Young children may be able to recite numbers but they may not be able to count, and they probably have a better understanding of ordinality at age four or five years than of cardinality.

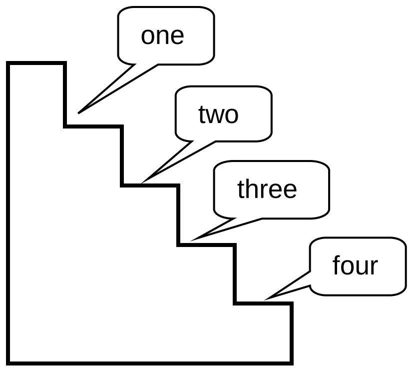
Let me explain: In coming to make sense of the world, young children associate name tags with objects. For example:

☎ this is a telephone
✐ this is a pen
✂ this is a pair of scissors

Each word describes a particular object. I can move them round and alter the order but the name tag moves with it.

✂ this is a pair of scissors
☎ this is a telephone
✐ this is a pen

Earliest counting will associate the number-name tags with the objects. The scissors will be number one, the telephone, number two, and the pen, number three. Watch a young child 'counting' stairs. With each step the child will say a number:



The number-name tags are associated with the order in which objects or events are encountered. For example, 'three' refers to a particular stair, and, at this point is unrelated to a quantity. Children are using the words 'one', 'two' and 'three' to describe an ordering. Some would describe this as an ordinal understanding. The words 'first', 'second' and 'third' etc. are known as the ordinals.

In everyday life, numbers are often used as names to describe a position in an order, such as table numbers in a restaurant, or seats in a cinema. Numbers on buses are used to describe particular routes. In these cases numbers are unrelated to the idea of a quantity. Knowing number names, recognising them expressed as numerals and knowing their order are some of the earliest points to be taught. Number rhymes, with appropriate displays, can help establish this knowledge.

Understanding that the last number in the count describes a unique property of the whole set of elements being counted is a concept that requires teaching and practise. Put yourself in the child's position:

Having become accustomed to using names to describe objects, they are now confronted with the proposition that the last name said, describes a property of the whole set. It is like saying that in the case of

✂ this is a pair of scissors
☎ this is a telephone
✎ this is a pen

that the word 'pen' has a meaning for all objects described.

✂ one
☎ two
✎ three

In the case of using number names, then the 'last' number in the count does have a meaning for all objects in the set. The meaning is in terms of the quantity of objects and is known as the cardinal value of the set. Numbers being used to establish the quantity in a set, or to answer the question 'how many?' are known as cardinals.

In preparing for the concept of quantity, teachers will be developing an understanding of 'more than', 'less than' and 'the same number as'. To embed the concept of quantity, pupils need to practise counting different objects and objects presented in different configurations. They need to be helped to understand that the last number in the count describes the quantity of objects counted. They need to know that if they count seven objects in one set, and count seven objects in another set, then there are the same number of objects in each set: the earliest notion of numerical equivalence. They need to know that if they count five objects in one set and count seven in another, then there are more in the second set than in the first, and less in the first set than the second.

A security with counting underpins a security with calculating. Check out how secure pupils are with their counting with this activity:

○ ○ ○ ⊙ ○ ○

Ask a four or five year old to count the counters above. Where did they start? Where did they finish? Ask them to make the 'dotted counter' number two in the count, or number six? What happened?

When pupils recognise that the order of the count and the configuration of the counters are independent from the cardinal value of the set then the concepts of cardinality and conservation are beginning to be established. Pupils are now ready to solve simple problems adding and subtracting counters or other objects. They are ready to progress to 'how many more?' and 'how many less?' and be gradually introduced to the symbolic representation of addition as '+' and subtraction as '–'. Having arrived at this point, they now see a number as representing a quantity. Early calculations in adding and subtracting to 10 can capitalise on representing a number as a quantity of counters. And this is effective. However, it is likely that calculations such as 3+4 or 6 - 2 will be undertaken as a count.

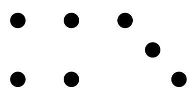
Building on extensive practice using practical apparatus and recording results, there is a need to prepare for these calculations to be undertaken mentally.

5. EARLY CALCULATIONS

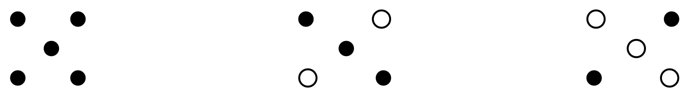
Providing images to work mentally with numbers to ten.

The concrete experiences offered to pupils must provide them with images which they can transfer into their minds. Images which can be imagined. Effective teaching of arithmetic centres on providing those concrete experiences, which can be translated into mental images, AND HELPING PUPILS MAKE THOSE TRANSITIONS. If we do not do this, young people (and old) will be forever relying on counting!!

When it comes to performing these simple calculations mentally, what images do we provide for pupils? Some pupils say that they 'see' counters in a row, but their recognition of a row of nine or ten is not instantaneous. Other pupils 'see' digits on two hands, whilst others 'see' dot configurations as on dice or dominoes. In these images, pupils instantly recognise the quantity.



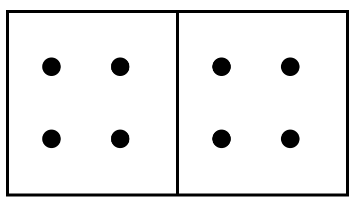
Both 4 + 3 = 7 and 3 + 4 = 7 can be seen and recognised, and 7 - 3 = 4 and 7 - 4 = 3 can also be seen in the same image.



5 - 2 = 3 can be seen and recognised, and 3 + 2 =5 and 2 + 3 = 5 can also be seen in the same image.

These images are rich in that they support the development of the concept of inverse, as Geoff Faux would argue, one of the big ideas in mathematics. Teaching addition and subtraction simultaneously is both sensible and efficient.

Doubling and halving can also be illustrated using the domino image, such as the one below, which shows double four as eight, and half eight as four:



With numbers to 10, a geometric array can <u>represent</u> a number as a quantity and provide a mental image when calculating. One might call this 'proto-calculation'. Given that any calculation can be undertaken as a count (for example, 4 x 8 as counting four rows of eight objects), there is a spectrum from counting to calculating, where efficient methods and routines would be placed at the calculation end.

The spectrum from counting to calculating

----------→----------→----------→----------→----------→----------→----------→----------

Counting
Calculating

In the examples above the count has been replaced with the <u>recognition of a quantity</u> in the form of a mental image. But, recognition of number quantity arrays becomes increasingly difficult as the quantity increases. I am still confused by my double-twelve set of dominoes.

6. COUNTING BEYOND TEN

Regularity of the number system. Irregularity of number names. Teaching so that regularity is understood.

And counting beyond ten presents particular problems. One of the features of our number system is its perfect regularity. We use this regularity when counting and calculating with larger numbers. We know that the numbers which follow 468 are 469, 470 and 471, not because someone has told us, but because we know how the number system works. The English number names, on the other hand, are not perfectly regular, and the first name irregularities are encountered in the second decade, that is from ten to twenty. Encountering this irregularity so soon in our count has the potential to confuse young learners by hiding the regularity of the number system itself. Our attention has recently been drawn to the success of Japanese pupils, and how they make more rapid progress in number than English pupils. A feature of the Japanese language is that the number names reflect the regularity of the number system.

THE JAPANESE NUMBERS

	1	ichi	(itchy)
	2	ni	(knee)
	3	san	(sun)
	4	shi	(she)
	5	go	(ready, steady, **go**)
	6	roku	(**rock** and roll)
	7	nana	(ba**nana**)
	8	hachi	(hutch)
	9	ku	(cuc**koo**)
	10	ju	(**ju**ice)
e.g.	11	ju-itchi	ten (and) one
	13	ju-san	ten (and) three
	20	ni-ju	two (times) ten
	40	shi-ju	four (times) ten
	57	go-ju-nana	five (times) ten (and) seven

Now try counting in Japanese!

In teaching pupils to count beyond ten, I have been impressed by the success of methods which emphasise the regularity of the numbers. For example, with 4 and 5 year old pupils who were secure in counting to ten, I recently saw a teacher prompting a chanted count by saying:
"Ten and one, ten and two, ten and three, ten and ------, ten and ------, ……."
Pupils were being helped to hear the repeating cycle of the units and link this to what they already knew. At the same time the teacher was showing the pupils cards, often called place-value cards, which illustrated how the number was formed.

> Place-value cards are simply pieces of card with numbers written on them. There will be nine 'unit' cards, each shaped as a square. There will be nine 'tens' cards each shaped as a rectangle made up of two squares, rather like a domino. A full set is provided at the end of this book, in the section "Some Practical Teaching Aids".

$\boxed{1\,|\,0}$ $\boxed{1}$

$\boxed{2}$

$\boxed{3}$

The teacher held the ten-card in her left hand for the pupils to see. For "ten and one" she picked up the single one-card and put it on top of the zero of the ten. For "ten and two" she replaced the single one-card with the single two card, and so on. The everyday number names were given, but the main objective of the lesson was to help pupils count beyond ten, using "ten and five" instead of "fifteen".

How confusing it is for pupils to separate thirteen from thirty one, or fourteen from forty one. How many times have we all seen pupils reverse the digits when writing the number thirteen? And if pupils lose their confidence at this early stage then subsequent progress may be thwarted. The key issue here is how do we teach counting beyond ten so that the pattern and rhythm of the perfectly regular number system is made explicit. I will return to this matter later.

The place-value cards make explicit the important fact that, say, eighteen, is made up of a ten and an eight, and NOT, a one and an eight.

7. ADDING AND SUBTRACTING BEYOND TEN

The image of the number line. Its use in teaching addition and subtraction of a number less than ten. A possible sequence for teaching.

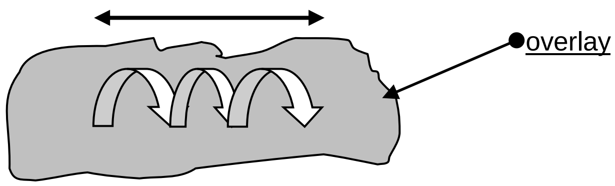
I mentioned earlier, that for mental calculation, pupils need to be offered images which can be imagined. For numbers to ten, common configurations can prove helpful. In preparing to mentally calculate beyond ten a more helpful image is needed, and this can be introduced alongside that above, to support working with numbers up to ten. A commonly used image is that of the number line.

```
---I-------I-------I-------I-------I-------I-------I-------I-------I-------I-------I-------I-------
 [1]     [2]     [3]     [4]     [5]     [6]     [7]     [8]     [9]     [10]    [11]    [12].
```

This is, again, a particularly rich image.

- As with the earlier image, it supports the development of the concept of <u>inverse</u>: The diagram shows how 4 + 3 = 7 and 7 - 3 = 4 are illustrated in the same image.
- It poses the notion of <u>infinity</u>, in that the line can extend indefinitely;
- It indicates the <u>relationship between numbers</u> in terms of their location on the number line;
- It illustrates a <u>structure</u> in the number system.
- It offers the prospect of <u>generalisation</u>, in that the arrow system on top can be shifted rightwards or leftwards to show all number pairs whose difference is 3; The focus has shifted from adding or subtracting 3 to/from a *particular* number to adding or subtracting 3 to/from *any* number;

Imagine that the triple jump arrow is drawn on a piece of tracing paper, or on an overhead transparency sheet. It can be only moved horizontally to the right or left. This 'overlay' represents add or subtract three, wherever it is placed on the number line.



The example below shows the overlay illustrated 9 + 3 = 12 or 12 – 9 = 3.

```
---I-------I-------I-------I-------I-------I-------I-------I-------I-------I-------I-------I-------
 [6]     [7]     [8]     [9]     [10]    [11]    [12]    [13]    [14]    [15]    [16]    [17].
```

Pupils, in using this approach, are still using counting, though in the calculation 4 + 3 pupils should be able to <u>recognise the location</u> of the first number rather than <u>count</u> it.

Pupils need to be helped to understand and use a number line. Clearly, their understanding of the order of numbers is going to be a starting point. Classroom displays can help reinforce number lines, ranging from number lines themselves, through calendars, to measuring scales, such as rulers. Measuring scales start to sow the seeds of thought of numbers between numbers. Rudy Rucker, in his book "Mind Tools" talks about number as being both "spotty" and "smooth". In thinking about number as answering the question "how many", then we are thinking of number as a discrete variable, and this opens the world of arithmetic. In thinking about number as answering the question "how much", then we are thinking about number as a continuous variable, and this opens the world of length and geometry. This duality in number can help in developing skills and understanding in both arithmetic and geometry, but it helps if it is recognised and acknowledged. A number line with numbers in 'squares' or on 'lines' can help with counting, but measuring requires numbers to be on the 'lines'.

I have seen productive lessons with five year olds, where they have been given blank number lines and cards with numbers on. Their task is to place the cards on the number line in the appropriate locations. Sometimes, one number is already written on the number line. This sort of activity can consolidate and measure pupils' understanding of ordering numbers to twenty and beyond.

I have seen some good whole-class teaching where the teacher has a large number line at the front and pupils are invited to count forwards or backwards from a given number. I have noticed that when using a number line for adding and subtracting, pupils initially count the numbers rather than the spaces arriving at an incorrect answer. The number line is only an image and pupils need to be taught how to use it appropriately.

A possible teaching sequence using number lines is outlined below:

Using the number line to teach addition and subtraction to twenty

Point to a number.
Pupils say the number.
Say "add 3".
Move the pointer 3 spaces, pupils count 1, 2, 3.
Pupils say the end number.

Repeat with different start numbers and different increments.

Introduce subtraction as reverse addition.
Point to 12
Add 4
Move pointer 4 spaces, pupils count 1, 2, 3, 4
Pupils say the end number, 16.
Point again to end number, 16. Pupils say 16.
Say "subtract 4".
Move the pointer 4 spaces back, pupils count 1, 2, 3, 4.
Pupils say the end number 12.

Discuss

Repeat with different start numbers, different increments, forwards and backwards.

Invite pupils to use the pointer at the front of the class.
Introduce the word "minus" as an alternative to "subtract".

Introduce more complex sequences:
e.g.
12 add 5 subtract 3;
14 add 3 subtract 5
10 subtract 4 add 4
11 subtract 3 add 2 etc

Discuss results:

Is the answer bigger or smaller than my start number?
Why?

Write down results:
e.g.
12 + 3 = 15
9 + 4 - 4 = 9

9 - 4 + 4 = 9 etc.

Introduce simple problems:
e.g.
What do I add to 6 to make 15?
What do I subtract from 14 to make 12?

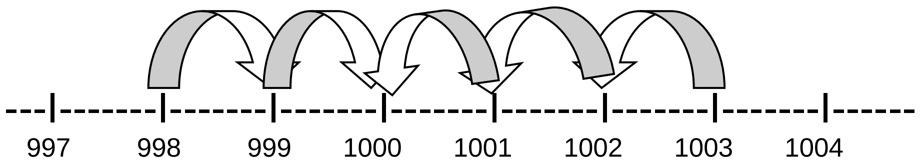
Focus on the language "add to" and "subtract from"

Introduce
☐ + 5 = 14
15 - ☐ = 12
16 - ☐ + 5 = 10   etc

Discuss and help pupils develop and articulate methods.

The image of the number line always starting at one or zero, becomes unwieldy as larger numbers are encountered, though number line segments will remain powerful and helpful images.

For example, the calculation 1003 - 998 is straightforward if the image below can be brought to mind.



Pupils who can calculate 1003 - 998 by mentally seeing and adding 2 to 3 are at an advantage over those who are dependent on only using a 'written method', such as decomposition.

8. COUNTING TO A HUNDRED

Providing an image of numbers to a hundred. Some 'embedding' activities for pupils. Using place-value cards as a teaching aid.

In order to extend the range of numbers with which to calculate, it is important that pupils can count within this extended range and know how the numbers relate to each other.

The number square can be introduced alongside the number line in preparation for working with numbers to 100. Pupils may make their own squares by cutting the number line into segments. Notice that the numbers for this exercise are placed within the squares. This geometric array of numbers reinforces the idea of regularity in the number system, continues to suggest notions of infinity and supports the understanding of inverse. The big ideas, proposed by Geoff Faux, can, in this image, be made explicit to pupils.

The number square drawn below is configured in its most common way. Many texts, however, start the count at one rather than zero. In my view, the structure is better illustrated by starting at zero, allowing single digit numbers to be located in the first row and double digit numbers starting in the second row. Some of my colleagues feel uncomfortable with an image where one moves 'down' to increase, and they argue that the number square should be configured with the smaller numbers at the bottom. I leave it to the reader to take their pick. The objective is to provide an image which pupils can readily imagine.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

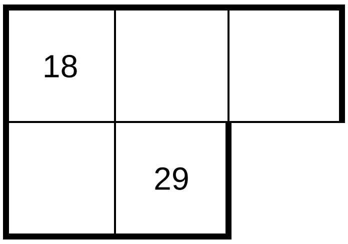
Counting in tens and multiples of ten now have a geometric significance in terms of vertical shifts. Combined with horizontal shifts for counting units, an image is presented which supports the development of methods for addition and subtraction of numbers to 100.

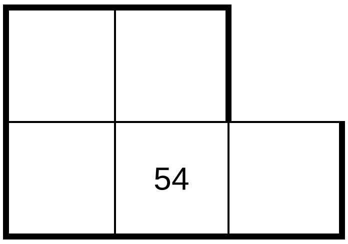
Teachers will need to ensure that pupils can construct this image before using it to calculate. I have seen some good lesson-starts where pupils are given cards from a pack of cards numbered zero to ninety nine. At the front of the classroom is a blank number square, with perhaps up to half a dozen numbers already located. Pupils are invited to put their cards into the appropriate spaces. A magnetic board is particularly effective.

There are many activities, in addition to that above, which help pupils fix their image of the number square. Three examples are given below:

1. Cut the square into a set of pieces – L-shapes, Π-shapes, + shapes, □-shapes etc. Mix them up and ask pupils to put the jigsaw together.

2. Take some of the jigsaw pieces and erase the numbers from some of the squares. Ask pupils to identify the missing numbers. (see below)





3. Ask pupils to close their eyes and to imagine the number square. Start at, say, 35. Now move down two places and then move three places to the left. Where are you now? Etc.

Counting can be helped by using an extended set of the place-value cards used ealier.

10	60	1	6
20	70	2	7
30	80	3	8
40	90	4	9
50		5	

The teacher can use these cards to explain how, say fifty two, is made up of a fifty and a two. In a lesson I taught recently, each pupil had their own personal set of place-value cards and I had a large number square in the front of the classroom. I pointed to a number on the square; the pupils said the number, selected the appropriate place-value cards and held them up. We extended the activity to showing one more and one less, and then ten more and ten less.

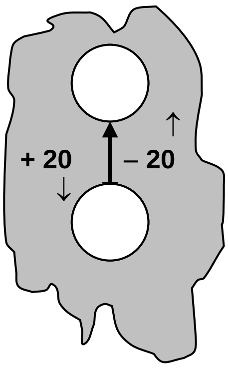
9. ADDING AND SUBTRACTING TO A HUNDRED

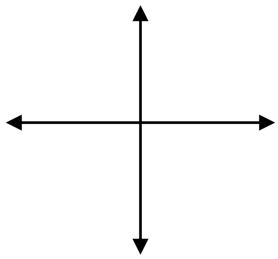
Using the number square as a teaching aid. A possible teaching sequence. Some limitations of the image. Some practise exercises for pupils.

Having become confident with the image, pupils need to develop the skill of adding tens, and then, multiples of ten, initially to multiples of ten (for example, 30 + 20), and then to any number (for example 30 + 57). The number square can be used as a teaching aid, by using ‘overlays’ to illustrate the geometric transformation. The number square can be used as a learning aid by providing an image which can be imagined. The overlay could be tracing paper for the pupils or an overhead transparency for the teacher. I have used both. The rule of this game is that the overlay can be moved only horizontally or vertically.

Superimposed on the number square below is an overlay which represents both “add 20” and “subtract 20”. Wherever I shift this overlay on the square, the operations remain valid. Pupils can practise this using their own number squares and tracing paper. Indeed, they may investigate all number pairs whose difference is 20.

The overlay will look like this:





0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

The square above shows 30 + 20 = 50, 50 – 20 = 30, 3 + 20 = 23, 23 – 20 = 3, 47 + 20 = 67 etc.

Having taught how to add or subtract multiples of ten to or from any number, the next step is teach how to add or subtract any pair of double digit numbers. The example below illustrates an overlay for adding or subtracting 26.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

By sliding this overlay vertically and horizontally around the square, a range of numbers, whose difference is 26 can be found. This starts to provide a powerful image for mentally adding and subtracting any double digit numbers.

Given that a pupil can mentally subtract a multiple of ten from any number, for example, 76 – 40, it is possible to "slide" any double digit subtraction into a position such that the number being subtracted is a multiple of ten.

For example, 73 – 28 is equivalent to 74 – 29, which is equivalent to 75 – 30, which I know is 45.
Or, 82 – 56 = 83 – 57 = 84 – 58 = 85 – 59 = 86 – 60 = 26
Or, 71 – 37 = 74 – 40 (by sliding three places or by adding three to both numbers) = 34.

This image is very powerful and prevents dependence on tricks such as 'borrowing and paying back', which are not always embedded in the structure of number. This image above addresses the real circumstance of equal addition.

Over dependence on a single image is precarious in that it may not be appropriate for all pupils. At the same time as using the number square to develop methods of mental calculation, the place-value cards, mentioned above, can be used to reinforce the composition of the numbers. For example, in the calculation 82 – 56,

$\boxed{8\,2} \quad - \quad \boxed{5\,6}$

Can be shown as:

$\boxed{8\,0} \quad - \quad \boxed{5\,0}$
$+ \qquad\qquad +$
$\boxed{2} \quad - \quad \boxed{6}$

Which can be shown as:

$\boxed{7\,0} \quad - \quad \boxed{5\,0}$
$+ \qquad\qquad +$
$\boxed{1\,0} \quad - \quad \boxed{6}$
$+$
$\boxed{2}$.

and/or

$$\begin{array}{ccc} \boxed{7\;\vdots\;0} & - & \boxed{5\;\vdots\;0} \\ + & & + \\ \boxed{1\;\vdots\;2} & - & \boxed{6} \end{array}$$

A possible teaching sequence is outlined below:

Using the number square to add and subtract numbers to 100

Link to the work using the number line, counting on in threes etc.
Show how the number square is related to the number line by cutting out and re-assembling.

<u>Adding tens</u>

Point to a number.
Pupils say the number.
Say “add ten”.
Pupils may initially count in ones. Through discussion, establish the fact that adding ten is a “move down”.
Repeat, repeat, repeat with a range of numbers and add ten.
Write down some calculations
e.g.
34 + 10 = 44
28 + 10 = 38
17 + 10 = 27 etc

<u>Develop into “subtract 10”</u>

Develop into “add twenty, thirty, forty …” etc
and subtracting multiples of ten
and combinations of adding and subtracting multiples of ten
use overlay.

Record some of the results
e.g.
34 + 40 - 10 = 64 etc

Develop into “add eleven, twelve etc”
Point to square and show adding ten and adding one,
adding ten and adding two etc
use overlays
For 42 + 12, pupils chant with pointer, “42 add 10, 52; add 2, 54”

Develop into adding and subtracting 24, 32 etc.
Discuss the moves being made on the number square.
Use templates/overlays

Focus on addition cases where the sum of the units is greater than ten
26 + 45: 20 + 40 = 60, 6 + 5 = 11, 60 + 11 = 71; or 26 + 40 = 66, 66 + 5 = 71

Use pointer to show add 19, 29, 39, 28, 38 etc
Discuss the moves being made
Discuss how add 29 is the same as add thirty, subtract 1
Use overlays. Chant the steps.
Discuss how subtract 29 is the same as subtract thirty, add 1
Use overlays to illustrate inverse operations.

Focus on subtraction cases where the units of the subtracted number are greater than the units of the number being subtracted from.

e.g. 42 - 27 is the same as 43 - 28 = 44 - 29 = 45 - 30 = 15 (use overlay) *equal addition*
or = 40 - 25 = 39 – 24 = 15 *equal subtraction*
or = 30 + 12 -20 - 7 = 10 + 5 = 15 *decomposition*

Discuss methods.
Practise.

Like all images, the number square has limitations. The most obvious being that it runs out at 100. You may by now have encountered another limitation. In exploring all the pairs of numbers whose difference is 26, the overlay that was used above sometimes goes over the edge of the square.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

?

?

One way to resolve this issue is to construct a "double-armed" overlay, one of whose arms will always be within the square, but the solution which best serves to illustrate the structure of the number system is to extend the number square beyond its square boundary.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
0	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

?

-44	**-43**	**-42**	**-41**	**-40**	**-39**	**-38**	**-37**	**-36**	**-35**	**-34**	**-33**	**-32**	**-31**	**-30**	**-29**	**-28**
-34	**-33**	**-32**	**-31**	**-30**	**-29**	**-28**	**-27**	**-26**	**-25**	**-24**	**-23**	**-22**	**-21**	**-20**	**-19**	**-18**
-24	**-23**	**-22**	**-21**	**-20**	**-19**	**-18**	**-17**	**-16**	**-15**	**-14**	**-13**	**-12**	**-11**	**-10**	**-9**	**-8**
-14	**-13**	**-12**	**-11**	**-10**	**-9**	**-8**	**-7**	**-6**	**-5**	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**
-4	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
6	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**
16	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**
26	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**
36	**37**	**38**	**39**	**40**	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**
46	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**	**61**	**62**
56	**57**	**58**	**59**	**60**	**61**	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**	**71**	**72**
66	**67**	**68**	**69**	**70**	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**	**81**	**82**
76	**77**	**78**	**79**	**80**	**81**	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**	**91**	**92**
86	**87**	**88**	**89**	**90**	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**	**101**	**102**
96	**97**	**98**	**99**	**100**	**101**	**102**	**103**	**104**	**105**	**106**	**107**	**108**	**109**	**110**	**111**	**112**
106	**107**	**108**	**109**	**110**	**111**	**112**	**113**	**114**	**115**	**116**	**117**	**118**	**119**	**120**	**121**	**122**
116	**117**	**118**	**119**	**120**	**121**	**122**	**123**	**124**	**125**	**126**	**127**	**128**	**129**	**130**	**131**	**132**

Not only is the structure of the number system revealed in this illustration, but also the notion of a structure set within an infinity. And notice the possibility of moving into negative numbers at this point.

Pupils need to practise what they have been taught. The intention is that they should be able to add and subtract mentally with numbers to a hundred.

Some practice exercises

A quick activity is to go round the class asking pupils to give the "hundred complement" of a number. In other words, given a number, the pupil has to say what number would need to be added to the given number to make 100. For example, the number 73 should elicit the response 27. This can be extended to give the complement to other numbers. For example, write 60 on the board, then give pupils numbers up to sixty and ask them to give its complement. "Twenty seven" would expect the reply "thirty three".

Another activity is to create a "round robin" of calculations, such that the answer to one question is the first number in the next question. I have listed a small set below to illustrate the activity. Each calculation is written on a card. The cards are shuffled and given to the pupils. Nominate one pupil to start. All pupils will have to solve each calculation in order to see if the answer is the first number on their card. The questions can be pitched at an appropriate level of difficulty.
For example, reading along the rows:

14 + 7	21 +4	25 + 30	55 – 6	49 – 10
39 + 30	69 – 4	65 + 20	85 – 5	80 – 40
40 + 3	43 + 4	47 + 40	87 – 20	67 – 6
61 – 30	31 + 40	71 +11	82 – 60	22 – 8

10. COUNTING BEYOND A HUNDRED

A new image – the Gattegno Chart. Confronting more number-name irregularities. Using the chart and place-value cards as teaching aids.

In working with numbers beyond a hundred, it remains critically important that pupils have a sense of how numbers the numbers they are working with relate to each other. For example, pupils are ready to work on a calculation such as 457 + 235 when they know which numbers live next to 457, what is 10 or 100 more, or less, than 457, and when they know the answer is just a bit short of 700. In other words, know the numbers before calculating with them.

A helpful image for working with the full range of numbers, is known as the Gattegno "tens chart".

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**
100	**200**	**300**	**400**	**500**	**600**	**700**	**800**	**900**
1 000	**2 000**	**3 000**	**4 000**	**5 000**	**6 000**	**7 000**	**8 000**	**9 000**
10 000	**20 000**	**30 000**	**40 000**	**50 000**	**60 000**	**70 000**	**80 000**	**90 000**
100 000	**200 000**	**300 000**	**400 000**	**500 000**	**600 000**	**700 000**	**800 000**	**900 000**

I have seen this used in classrooms as a wall display and used by teachers to teach pupils both number names and number structure. It is most effective when coupled with the use of the place-value cards described above, but this time extended to include hundreds. I have set out some examples below.

The teacher points to 40; pupils chant "forty"; teacher points to points to 5; pupils chant "five". Pupils may select their forty-card and their five card and place the five card over the zero of the forty, illustrating the number 45. This can be practised at length, and sequences of consecutive numbers may be derived.

This approach powerfully confronts the issue of the irregularity of number words, notice:

Nine	ninety	
Eight	eighty	
Seven	seventy	
Six	sixty	
But:		
Five	fifty	and not five-ty
Four	forty	and not four-ty
Three	thirty	and not three-ty
Two	twenty	and not two-ty
One	ten	and not one-ty.

It is important that pupils recognise these word irregularities at the same time, recognising the number regularity.

And it becomes even more irregular when the chanting exercise described above deals with the second decade.

Teacher points to 20; pupils chant “twenty”; teacher points to 1; pupils chant “one”, forming the number 21. But, teacher points to 10 and pupils now need to wait to see if the teacher is going to point to a unit, say 3, before they can form the number “thirteen”. I know that some teachers accept, as a transitional arrangement, “onety three” or “tenty three”, and there is no evidence that these pupils have been damaged by the experience. I have mentioned this approach on page 12.

This point and chant approach, with the whole class, can now extend to any number.
Teacher points to 300; pupils chant “three hundred”; teacher points to 60; pupils chant “sixty”; teacher points to 4; pupils chant “four”. They have built the number 364, which they may reinforce by building with their number cards:

3	0	0					
	6	0	→		3	6	4
		4					

Even here, there is an irregularity. We do not actually say “three hundred sixty four”, but we say “three hundred **and** sixty four”. Again, pupils’ attention needs to be brought to this matter. Whenever moving from the third to the second row on the chart, the word “and” is introduced.

I leave it to the reader to see how this chart can be used to understand numbers up to a million. I will return to this image when considering multiplication. Notice that the space between thousands and hundreds echoes the space inserted when writing these numbers.

I have constantly been referring to images which pupils can imagine in order to work with numbers in their heads. Images I have selected are those which illustrate the structure of number and make evident those big ideas of mathematics listed at the beginning of this book. In this way, the essence of mathematics is being made accessible to pupils. The more structure is understood, the less is needed to be committed to memory. I find the aphorism below, coined by Dave Hewitt, particularly helpful.

Relying on memory, brings with it the risk of forgetting

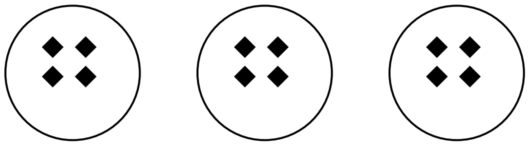
Remember the message from the activity at the end of Section 3.

11. MULTIPLICATION AND DIVISION

Teaching the concept of multiplication and division. Teaching tables – some suggested sequences. Practise exercises for pupils. Multiplying by powers of ten using the Gattegno chart. A resumé of what can be done mentally.

The next section addresses the teaching of multiplication and division, focusing on the acquisition of multiplication and division facts with numbers up to a hundred. Just as it is logical and efficient to teach addition and subtraction together, so the same applies to multiplication and division. The statement $4 \times 3 = 12$ carries with it the statement $12 \div 3 = 4$. Indeed, because 4×3 gives the same product as 3×4, then $12 \div 4 = 3$ may be implied. These interrelationships need to be taught.

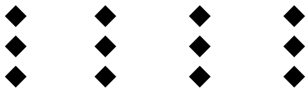
4 x 3 can be demonstrated as three 'lots' of four, thus:



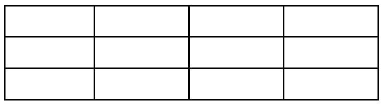
This may well be a helpful starting image, which links the numbers to quantities of objects. And, pupils can use their counting skills to interpret the concept of multiplication as repeated addition, or division as equal shares.

The word 'multiply' has an interesting derivation. The word 'ply' derives from a Scottish word meaning fold or layer. The Old French meaning related to a twist, hence the use of the word 'ply' in the context of wool: two-ply, three ply etc. The notion of multiply as multi-layered provides a helpful geometric image.

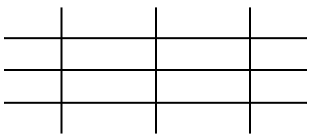
4 x 3 can be demonstrated as three layers of four, thus:



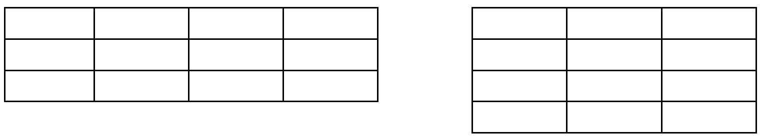
or using the rectangles:



or using the intersections of the lines:



Now the relationship between 4 x 3 and 3 x 4 can be illustrated geometrically:



But, back to teaching tables. Having in your minds the notion of economic learning, then, knowing tables should not depend on memorising a hundred different facts.

From an early age pupils will be developing ideas of doubling and halving. Using dominoes and pairs of dice can help in providing appropriate experiences. Doubling will initially be understood as adding a number to itself, and practice will be needed to develop fluency in this skill. Halving is often introduced as halving a physical entity, such as a cake or geometric shape. It is important that, at the same time, the halving of a quantity is taught. In this way, understanding the relationship between doubling and halving can be secured.

Using doubling, some of the multiplication tables are now accessible. The order in which the tables are taught can capitalise on "gaining a lot from a little".

For example 3 x 2 is double 3, which is 6. Now, if I know that 3 x 2 = 6, then, because 4 is double 2, then 3 x 4 will be double 3 x 2, which gives me double 6, which is 12. And if I know that 3 x 4 is 12, then I can double to get 3 x 8 = 24. By treating the 2, 4 and 8 times tables together, the doubling relationship can be made explicit and exploited.

	1	2	3	4	5	6	7	8	9	10
X	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
X	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
X	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

A blank table, such as the one above, can be helpful in building up particular families of multiples, and, through teaching, making explicit the relationship between them. For example:

	1	2	3	4	5	6	7	8	9	10
X 2	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
X 4	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
X 8	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

The teacher can point to a series of locations, perhaps along a row, or down a column, and pupils chant their responses. Or a pupil can work on completing the table themselves.

Another family of multiples includes the multiples of three and the multiples of six. Multiples of three may be seen as tripling, or as a double multiple added to a single multiple. For example, 7 x 3 is double seven added to a single seven. The table below may help in developing this understanding:

	1	2	3	4	5	6	7	8	9	10
X 2	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
X 3	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
X 6	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

Just as adding and subtracting tens, and multiples of ten, extends addition and subtraction skills, so multiplying and dividing by ten, and multiples of ten extends multiplication and division skills. Because our number system is based on ten, multiples of ten are easily identified. Indeed, pupils can recite multiples of ten before those of smaller numbers. They will have developed these skills in learning to count, and the number squares used earlier will have reinforced their confidence.

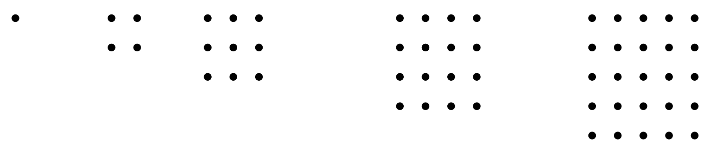
Because multiplying by five gives half the product of multiplying by ten, I have identified another family of multiples:

	1	2	3	4	5	6	7	8	9	10
X 5	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
X 10	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

In teaching this pair of tables, I would point to the tens first and then back to the fives. For example, Point to 3 x 10; pupils respond “thirty”, then point to 3 x 5, which is half 3 x 10, so pupils respond “fifteen”.

Another family could be the ten and nine multiples, where the multiples of nine are derived from subtracting a single multiple from the multiple of ten. For example, 8 x 10 is 80, so 8 x 9 is 8 x 10 – 8 x 1 giving the value 72.

Using the principle of commutativity of multiplication, (by which, I mean a x b is equivalent to b x a), pupils are just about ready to complete a ten by ten table square. The only multiplication not dealt with is 7 x 7. This can be derived by adding or subtracting a multiple to 7 x 6 or 7 x 8 or by adding or subtracting a seven to 6 x 7 or 8 x 7. Or teachers may have done some work exploring the square numbers:



etc

1 x 1 = 1   2 x 2 = 4   3 x 3 = 9   4 x 4 =16   5 x 5 = 25

Now the multiplication square can be completed, and all the other patterns and interrelationships explored. This can assist in developing rapid recall.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

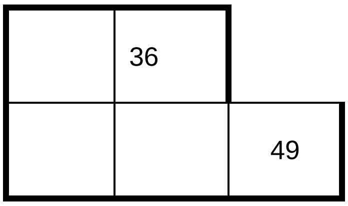
There are numerous publications which offer interesting activities for pupils to work on which involve the multiplication square. I offer four, the first three of which are similar to those related to the number square.

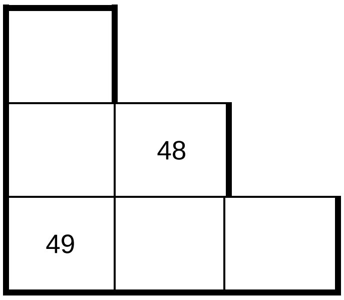
Some practice activities

Firstly, using cards with the numbers 1 to 100 written in them, deal them out to pupils and ask them to place them on a blank table square, or a square with only a few multiples located. There are some interesting and productive teaching points to be made about the cards which are not used, and cards which could be inserted in more than one location.

Secondly, cut up the multiplication square into 'jigsaw' pieces and invite pupils to put the jigsaw together. I have designed and used a large-scale square jigsaw which has the number square on one side and the table square on the other. This has proved a useful piece of apparatus. It has been interesting to listen to pupils' reasoning as they work out which way up each piece should go.

The third activity involves selecting pieces of the jigsaw, removing some of the numbers, and asking pupils what the numbers were. For example:





The fourth activity, encourages pupils to think as much about division as multiplication, and reinforces the understanding that each is the inverse of the other. In the example below, some multiples are given and pupils are challenged to find the appropriate factors to complete the multiplication grid:

X	?	?	?
?	8	12	16
?	12	18	24
?	24	36	48

The reader may wish to miss out some of the multiples.

Before leaving mental multiplication, I would like to return to multiplying by ten, and by powers of ten. The “Gattegno tens chart” can provide a useful image. I have taken out the gap between thousands and hundreds, as that was there to illustrate how numbers were written.

I have identified a number by putting a ring around each of its parts. In the example below, I have chosen 658. Notice that by moving the overlay up and down, I am illustrating multiplication and division by powers of ten

The table below illustrates 658 x 100 = 65 800, and
65 800 ÷ 100 = 658

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**
100	**200**	**300**	**400**	**500**	**600**	**700**	**800**	**900**
1 000	**2 000**	**3 000**	**4 000**	**5 000**	**6 000**	**7 000**	**8 000**	**9 000**
10 000	**20 000**	**30 000**	**40 000**	**50 000**	**60 000**	**70 000**	**80 000**	**90 000**
100 000	**200 000**	**300 000**	**400 000**	**500 000**	**600 000**	**700 000**	**800 000**	**900 000**

The table above can be used to illustrate and practice multiplication and division by powers of ten. As with the image of a 100 square, for certain calculations, the overlay may go beyond the boundary of the table. But, the table is only a subset of the infinity of numbers. By extending the table, the world of decimal numbers is revealed. Look at this!

0.01	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.1	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**
100	**200**	**300**	**400**	**500**	**600**	**700**	**800**	**900**
1 000	**2 000**	**3 000**	**4 000**	**5 000**	**6 000**	**7 000**	**8 000**	**9 000**
10 000	**20 000**	**30 000**	**40 000**	**50 000**	**60 000**	**70 000**	**80 000**	**90 000**
100 000	**200 000**	**300000**	**400 000**	**500 000**	**600 000**	**700 000**	**800 000**	**900 000**

The table above illustrates:

35 x 10 000	=	350 000
35 ÷ 100	=	0.35
0.35 x 100	=	35
0.35 x 1 000 000	=	350 000
350 000 ÷ 10 000	=	35
350 000 ÷ 1 000 000	=	0.35

Knowing the structure of the number system, and having a useful image of it, allows multiplication and division of any number by powers of ten to be undertaken with ease. (e.g. x 10, x 100, x 1 000, ÷ 10, ÷ 100, ÷ 1000 etc.)

Couple this with the ability to multiply any pair of single digit numbers, and the possibility of multiplying and dividing by any multiple of any power of ten is now available. (e.g. x 20, x 500, x 3 000, ÷ 70, ÷ 500 ÷ 2 000 etc.)

Couple this with the ability to add and subtract numbers, and the possibility of multiplying and dividing by any number is now available.

But I am now at the point where I need a piece of paper to write down some of the interim steps in the calculations.

Before considering the necessity to write down interim steps in complex calculations, it is worth taking an inventory of what can be undertaken mentally:

The mental tool kit

- Adding and subtracting whole numbers up to a hundred.
- Adding and subtracting multiples of powers of ten.
- Multiplying any pair of numbers up to ten.
- Dividing any number to a hundred by one of its factors.
- Multiplying and dividing any number by a power of ten.

This is a powerful tool kit and pupils need to practise using it in as many different contexts as possible. One of the contexts is in the solution of more complex calculations.

The economy of learning, and the efficiency of teaching, which is gained by understanding the structure and make up of the number system cannot be underestimated. I was recently reminded of this when talking to a five year old who told me he knew that three and four made seven. I asked him what else he knew. He replied that he knew that three hundreds and four hundreds made seven hundreds, and that three thousands and four thousands made seven thousands. I asked him what three millions and four millions made. He quickly responded with seven millions.

This confirms that progression in learning is not always reflected in the textbooks that are used in schools, and that 27 + 18 may be a more challenging calculation than three million add four million. There are some serious messages here for textbook writers, and for teachers when they are planning for progression.

12 EXTENDING BEYOND THE COUNTING NUMBERS

Extending the number line and using it as a teaching aid to teach negative numbers, fractions and decimals. Using these images to add and subtract.

Some of the images presented earlier have over-spilled into negative numbers or into decimal fractions.

When using some of the images of our number system, extending the numbers beyond the counting numbers becomes natural and inviting. For example, when using a number line, such as the one below,

1	2	3	4	5	6	7	8	9	10	11	12 .

the question, "what comes next" is invited. And, by what comes next, I mean both rightwards and leftwards.

-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13 .

If pupils have had experience of reading and recording temperatures it is likely that they will have heard of negative numbers. It is important that they are taught where these numbers live in the structure and how they relate to the counting numbers.

When using the number line, such as the one below, again questions can be asked in relation to what numbers live in the gaps.

```
----|----*----*----*----*----*----*----*----*----*----|----*----*----*----*----*----*----*----*----*----|----*
    0                                                 1                                                 2
```

As pupils gain experience in measuring lengths and reading scales, they can be helped to see that any number they encounter has its place in the number structure.

For example,

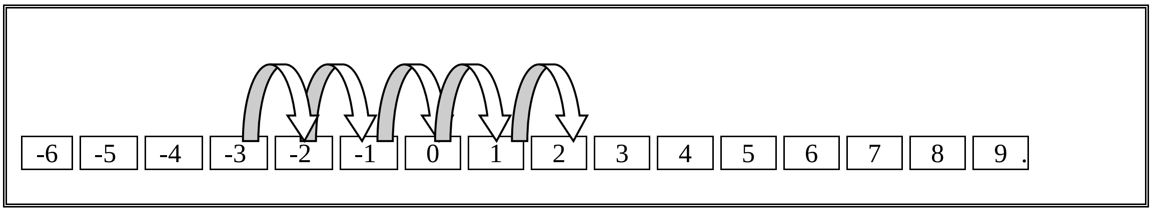
```
----|----*----*----*----*----*----*----*----*----*----|----*----*----*----*----*----*----*----*----*----|----*
    0           ¼           ½           ¾             1                        1½                       2
```

And another example,

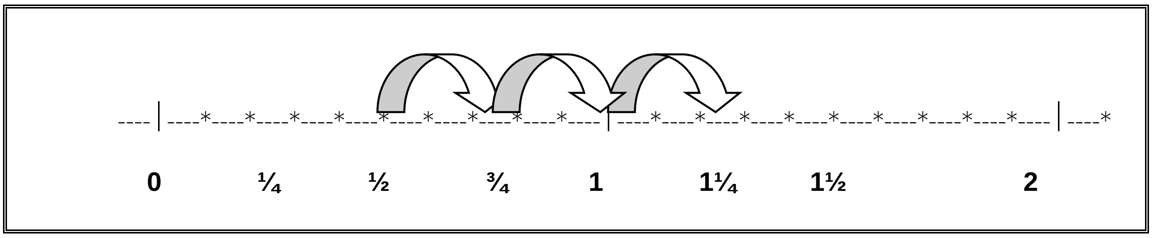
```
----|----*----*----*----*----*----*----*----*----*----|----*----*----*----*----*----*----*----*----*----|----*
    0  0.1  0.2  0.3  0.4  0.5  0.6  0.7  0.8  0.9    1  1.1  1.2  1.3  1.4  1.5  1.6  1.7  1.8  1.9    2
```

Methods used to teach mental addition and subtraction of whole numbers can now be used across the wider range of numbers.

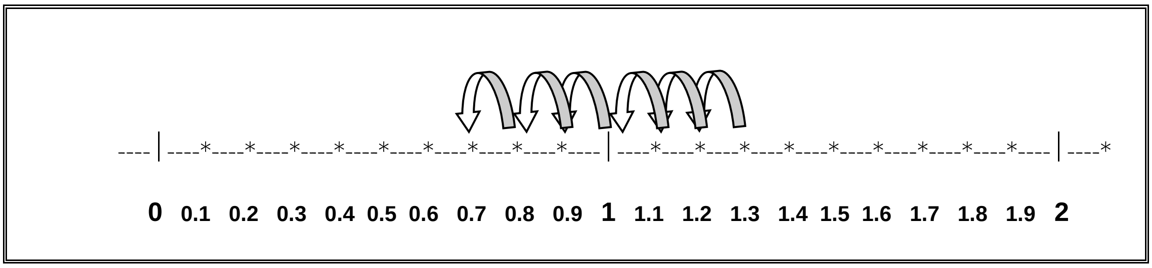
For example, -3 + 5 = 2 (or, 2 – 5 = -2)



And, another example, ½ + ¾ = 1¼ (or, 1¼ - ¾ = ½)



And another example, 1.3 – 0.6 = 0.7 (or, 0.7 + 0.6 = 1.3)



You have already seen how the number square can be extended to illustrate negative numbers, and how the "Gattegno tens chart" can be extended to illustrate decimal fractions.

The message is clear. There is a single number structure which is perfectly regular and accommodates all numbers. Understanding this structure is the key to learning how to calculate. Presenting appropriate images of this structure, and explaining particular transformations, is the key to teaching how to calculate.

13. WRITTEN CALCULATIONS (Part One)

The connection between written and mental calculation. Four examples with a commentary on strategies used.

> "When children obtain answers to sums and problems by mere mechanical routine, without knowing why they use the rule … they cannot be said to have been well versed in arithmetic."
> (An 1895 Report)
>
> "To succeed it is necessary rather to furnish occasions for them to exercise their own skill in performing examples rather than to give them rules. They should be allowed to pursue their own method first, and then should be made to observe and explain."
> (Intellectual Arithmetic 1840)

Mental methods depend on pupils having an understanding of what they are doing. I have argued that pupils can be helped, by having images of the number structure, on which they can make transformations appropriate to the calculation they are undertaking. I have also argued that these images need to be taught and the transformations explained and practised.

Written methods are not distinctive from this. Algorithmic icons, such as 'subtraction by decomposition', 'long multiplication' and 'long division' are often regarded as teaching topics in their own right and expected to be applied in even the most inappropriate circumstances.

I have tried to avoid making my point by citing some of the poor practice that I have observed, but I believe it may be helpful here. A teacher was helping a pupil who was struggling with the calculation 13 – 7 . It had been set out in writing as:

```
13
 7  -
__
__
```

Teacher:	You start off by saying three take away seven, well?
Pupil:	Well, you can't.
Teacher:	So? What do you do?
Pupil:	Um, you cross out the one and put it next to the three.
Teacher:	Good, well do it then. Now what have you got?
Pupil:	Thirteen take away seven.
Teacher:	Which is?
Pupil:	(Pause for thought) Six.

At this point I started to feel sorry for the pupil who could clearly undertake the calculation, but whose confidence in numbers, I suspect, was beginning to erode.
The conversation continued:

Teacher:	Now write the six down under the seven. Now what next?
Pupil:	Nothing take away nothing.
Teacher:	Which is?
Pupil:	Nothing. (and the pupil starts to write a zero in the tens column)
Teacher:	There's no need to do that, because it is nothing. Now try the next sum.

What was the purpose of this exercise? How did it build on the pupil's mental understanding. I propose the following principles should be applied when selecting methods for calculating:

Calculation methods should be:

- **Appropriate;**
- **Effective;**
- **Efficient.**

In the example given above, it would have been more appropriate to have visualised an image and applied a mental strategy to obtain the correct answer. Indeed, the conversation revealed that the pupil was quite capable of doing this.

In determining an appropriate method, the first consideration should be: "can I solve this mentally?". If not, then "can I break this up into parts which I can solve mentally? And what do I need to write down?" If this is still not possible, then "how can I use a calculator?"

The effectiveness of a method is to do with achieving an accurate solution and knowing it is correct.

The efficiency of a method is related to the number of steps taken to achieve an accurate solution. Using multiplication is more efficient than repeated addition; using division is more efficient than repeated subtraction. When calculating percentage increase, using a calculator, this can be done as a single multiplication, rather than a multiplication followed by addition.

For example, increase £34.50 by 17½%

34.5 X 1.175

is more efficient than

34.5 + 0.175 X 34.5

and this might be ineffective if the calculator does not have 'algebraic' logic. In other words, if the calculator adds 34.5 to 0.175 and then multiplies the result by 34.5 the answer will be wrong. A calculator with 'algebraic' logic will ensure that the 0.175 is multiplied by the second 34.5 before adding it to the first 34.5.

Pupils need to be helped to be discerning calculators, and they need to be taught how to apply the principles of appropriateness, effectiveness and efficiency.

Section 14 illustrates how efficiency may be taught by critically evaluating pupils' approaches to calculating methods. This centres around pupils discussing their methods with others. In Hungary, it is typical practice for pupils to work on a problem and then to be invited by the teacher to work their problem on the board, in front of the whole class. Other pupils are invited to comment on the method used, and whole-class discussion will lead to refinement and greater efficiency. Mathematicians call this mathematical elegance.

This section looks at four such problems.

What follows are four questions, which I presented to some Junior school age pupils. I asked them to think carefully about each problem and to use methods to solve each problem, which they were confident would give the right answer. They were invited to write down any steps which they needed to. They were given no help during the exercise. Before you look at their responses you are invited to do the same. I have faithfully reproduced their responses, warts and all!

Problem 1.

In summer I am going to take the 274 pupils and staff in a school on a trip to Bamburgh. The bus company tells me that each bus will hold 62 passengers. How many buses will I need?

1 Zoë 11yr

62) 274, with 421 written above

62) 200.76, with 2 written above

38 + 38 = 76

62 + 62 = 124
124 + 62 = 186
186 + 62 = 248

274 − 248 = 026

5 buses

Bus 1 = 62 people
Bus 2 = 62 people
Bus 3 = 62 people
Bus 4 = 62 people
Bus 5 = 24 people

yu:utu

62
2 124
4 248
5 50

You will ~~na~~ need 5 buses.

Claire
No. 1.

274 − 62 = 212
212 − 62 = 150
150 − 62 = 88
88 − 62 = 26

You need four buses and
there will be 26 left

Notice Zoë's work. She knew that she wanted to find how many 'sixty-twos' there were in 274, and that this could be achieved through dividing. But, she has forgotten the 'method'. In fact, she has tried the 'method' and knows that the answer it has given is unreasonable. "It couldn't possibly be twenty one". Her second attempt is again 'long division', and she tries to find out how many 62s there are in 200. Perhaps this is easier than finding how many 62s in 274. Her attempt shows she has found that there are two 62s in 200 and that there is a remainder of 76. She mentally halves the 76 (and checks it) to see how these could be shared out. She has arrived at an impasse and decides to start again. This time she adds groups of 62 until she arrives at 248. (four groups counted so far) She senses she is close and subtracts the 248 from 274 to get 26, which will require another bus. She records her findings by saying that five buses are needed, and shows how many would travel in each bus. She mentally checks the total and knows that she has a reasonable and accurate answer.

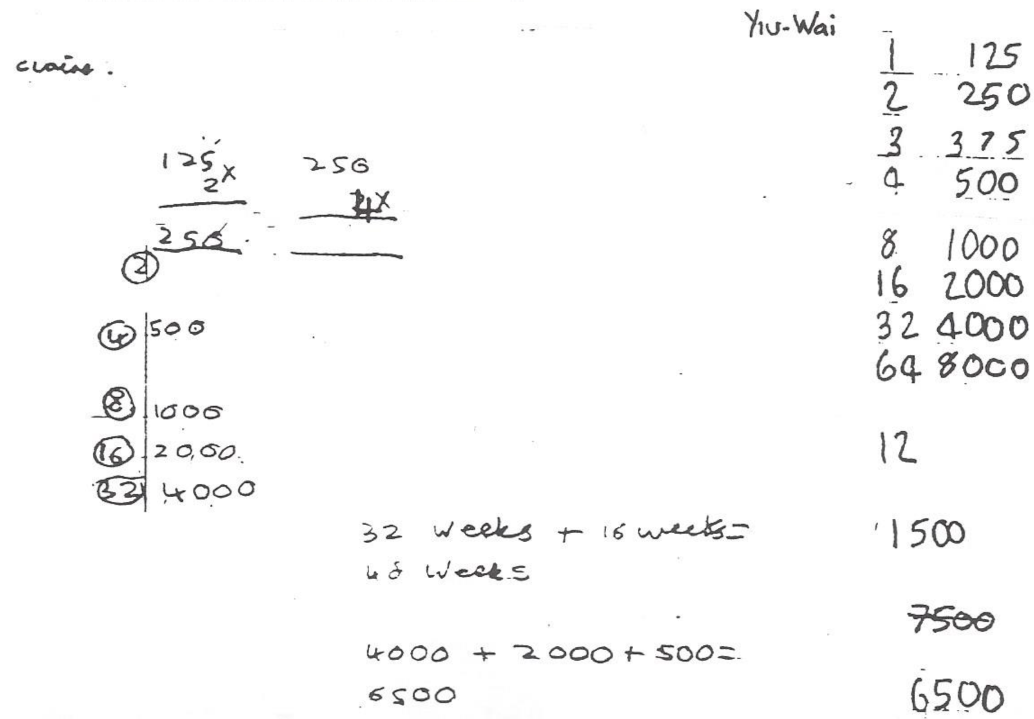
Notice Yiu-Wai's work. He starts with 62, and doubles, and redoubles, recording how many multiples. At four he senses he is close, and, rather than calculate eight multiples, which he knows is too big, he starts to work out five multiples. He writes down zero units and can see in his head that the answer is 5 (which, in his excitement, he writes next to the zero). He has finished and clearly communicates his result.

Notice Claire's response. The way that she sees the problem is expressed in the way she sets about solving it. The first bus arrives, sixty two people get on, and now there are two hundred and twelve left. The next bus arrives, sixty two get on, and so on. She clearly has a sense of economy! But, who am I to question a value judgement!! How would you assess her answer?

Both my wife and I have extensively researched how pupils attack this sort of problem, and these approaches are typical. I urge you to try it out with your pupils, but do make it clear that they are allowed to use any method which they know will produce an accurate answer. Remember the second principle: the method must be effective. Another approach, not exemplified here, is to make an informed guess and then check it out. For example, I think the answer is four, or five, or six, and then check by multiplying, and then modify the initial estimate accordingly. Or, perhaps ten sixty-twos are six hundred and twenty, half of which is three hundred and ten. It is what is in the **mental** tool kit that is so critical here.

Problem 2

Each week, I buy a packet of stamps. Each packet contains 125 stamps. How many stamps do I collect in a year (52 weeks)?



Zoë 11yr

125
52
250
6250
6500.

6500 stamps get collected in 1 year
(52 weeks

Notice Claire's work. She is starting from the bottom and working upwards. She starts by multiplying a hundred and twenty five by two, and correctly gets 250. She goes on to set out 250 by two, crosses this out and jumps to four. She has noticed something special about the pattern and realises she knows the answers. She reverts to simple recording her doubling and continues up to the fifth power of two (32). She goes on to add what 32 and 16 weeks are worth, and concludes by adding 500, which is what the additional four weeks are worth. She communicates her result.

Yiu-Wai's response is similar. He seems to be a 'doubler' by instinct. But look at his working. He doubles and redoubles up to the sixth power of two. But notice his tentative start, where he calculates the third multiple, and senses that he needs to move on. He finds that 64 multiples of 125 has overshot where he needs to be. His recording of 'twelve' indicates his awareness of an overshoot by twelve multiples. He writes down '1 500' which is the value of these twelve multiples (64 – 12 = 52) and then mentally subtracts the 1 500 from 8 000. His first shot, at 7 500, feels unreasonable and he self corrects to 6 500.

And Zoë, who sees the problem as multiplying 125 by 52, and uses a method in which she has confidence. (She later explains that she multiplies by 2 and then by 50, and then adds the two multiples together).

Problem 3

For a school fete, I have decided to run a comic stall. Each pupil will bring to school 24 comics. There are 357 pupils in the school. How many comics should be collected?

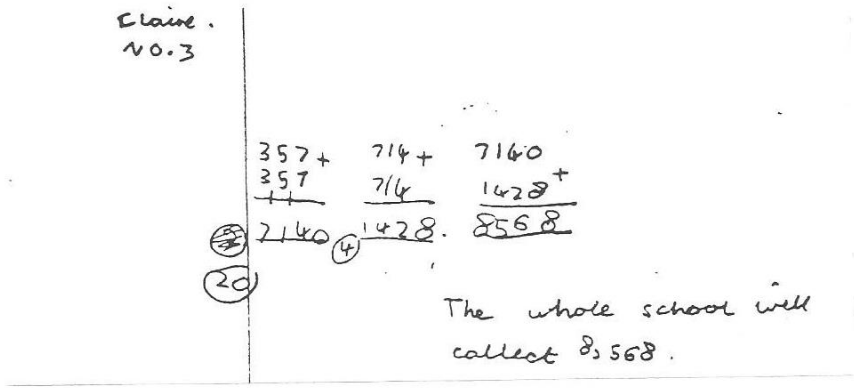
Zoë 11yr

100 lots of 24 = 2400
2400
+ 2400
7200

50 lots of 24 = 1200 half of 2400
7 lots of 24 = 24
× 7
168

1200
7200
+ 168
8568 = 8568

24 X 357
24 ×
8568



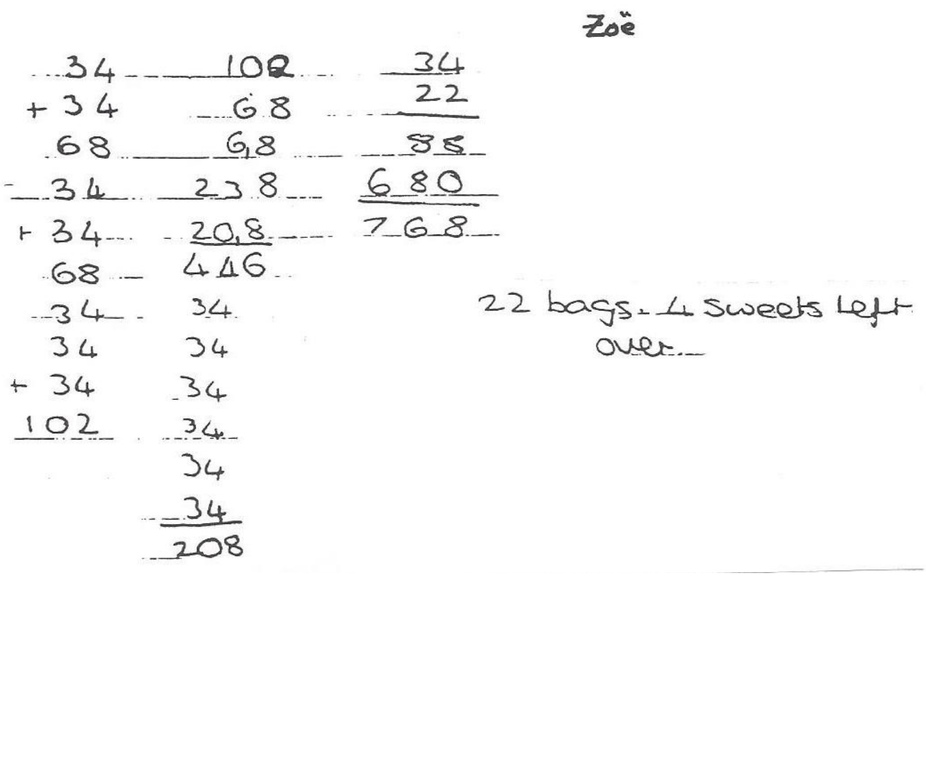
I am intrigued by Zoë's response. Remember, she appeared to be confident with the 'traditional long multiplication' algorithm in the previous example. Her response here shows her falling back to her level of confidence. She is confident with multiplying by powers of ten, but not by multiples of powers of ten. She multiplies by the second power of ten (100) and adds three lots. She knows that multiplying by 50 is half a multiple of 100 and then multiplies by seven. She adds her sub-totals together.

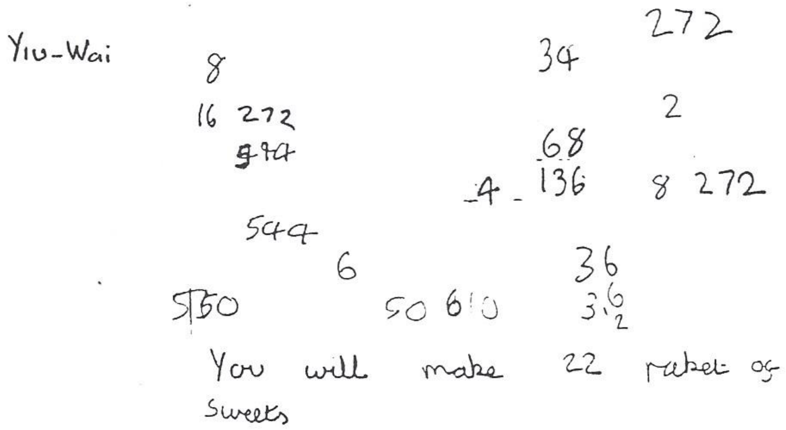
Yiu-Wai multiplies directly by twenty-four but uses his understanding of commutativity to change this to multiplying by the single digit. In other words, he multiplies the twenty-four by seven mentally, and confidently 'carries' a double-digit number.

Claire's approach is interesting. She again starts from the bottom and works upwards. She starts by doubling by adding the number to itself. She writes a 'two' to keep a record. She goes on to add the double multiple to itself, to get the fourth multiple, and makes a record.. She then sees that she is near a solution. She modifies the second multiple, to make it the second multiple of ten and modifies her recording log accordingly. She adds the two sub-totals together.

Problem 4

I am making bags of sweets, and each bag will have thirty four sweets in it. I bought a large box from the supermarket, which contains 760 sweets. How many bags can I make?




Yiu-Wai
8
16 272
~~5~~44
544
6
550
34
68
4 136
36
3 6 2
272
2
8 272
50 610
You will make 22 paket of sweets

Well! Zoë has started by grouping bags in twos. She realises this will take some time and this has driven her to investigate larger groupings. She finds a three-bag grouping, which makes 102, and goes on to add a three-bag grouping to two two-bag groupings. She makes an error in calculating a six-bag grouping, and writes down 208, instead of 204. With all these groupings she makes an estimate of 22, though her check is inaccurate, as she writes 88 as the product of 2 and 34. For some reason, she sticks with her estimate, but is inaccurate with the number of sweets left over.

Yiu-Wai uses doubling, though his recording is idiosyncratic. He finds two bags are worth 68, four bags are worth 136, eight bags are worth 272, and sixteen bags are worth 544. He goes on to estimate that 22 bags are required.

14. WRITTEN CALCULATIONS (Part Two)

Implications for teaching. The pit-falls of short-cuts. Teaching for efficiency. The place of traditional or stylised algorithms.

In solving problems, such as those above, the pupils have brought to bear:

- their understanding of the problem;
- a range of mental calculation strategies.

And they have written down;

- that which they need to, in order to log interim findings;
- the answer, in order to communicate their solution to the problem.

In the examples above, pupils were presented with problems and challenged to solve them using whatever tools in their mental toolkit they felt confident in using. They were engaging their minds with the problem. In the first problem, Zoë solved it by repeated addition, Yiu-Wai by doubling, and Claire by repeated subtraction. Their strategy for solution was based on how they "saw" the problem and what mental tools they could bring to bear.

The problem is not, by definition, "a division problem" or a "multiplication problem". In many cases, pupils may estimate an answer, use a calculation to check the estimate, and adapt the original estimate accordingly. Notice, in the first problem, how Zoë's change of approach was made when she saw that a method she was trying to remember (long division) gave her an answer of twenty one, which was, in her mind, clearly unreasonable. So, a change of tack onto a strategy she knew would work.

Sometimes, in desperation, teachers try to take the thinking out solving problems. They train pupils to look for key words and phrases, and to associate these with particular operations. For example,

The word(s)	*altogether*	will mean you have to	**add**	
	remainder		**divide**	
	groups of		**multiply**	
	difference between		**subtract**	etc.

Development of mathematical vocabulary **is** critically important, and is a gateway to understanding and the ability to think and reason. But, the one-to-one association of phrases with operations is not always helpful. I might see the difference between ninety eight and a hundred and one as three, because I add three onto ninety eight to make the hundred and one. I might see the remainder from sharing seventy six sweets equally amongst nine people as four, because I have multiplied nine by eight and subtracted the seventy two from the seventy six. Linking, too strictly, phrases to operations, can limit efficient use of mental insights and methods.

Again, where pupils are insecure with their mental tool kit, teachers provide a 'written method' which will always work.

> Taking the thinking out of solving problems and calculating fails **because** the thinking has been removed from the process, and has been replaced with a dependence on memory.

Mental strategy and 'written methods' are part of the same process. In a complex calculation I employ mental strategies, but I may need to jot down some intermediate steps because I cannot hold them in my head. These jottings are the first manifestations of my 'written method'. Separating mental methods from written methods can fracture pupils' progression in learning and undermine their confidence.

> What is written should reflect mental strategy and not determine it.

Teachers often express concern that, having taught the stylised algorithms for the four number operations, then many pupils have difficulty in applying them to the solution of problems. This takes me back to the ingredients of numeracy (pages 7 and 8) where I mention the importance of confidence. Solving problems is what numerate individuals should be able to do.

"The ability to solve problems is at the heart of mathematics. …. However, the solution of a mathematical problem cannot begin until the problem has been translated into mathematical terms. This first and essential step presents very great difficulties to many pupils – a fact which is too little appreciated. At each stage of the mathematics course the teacher needs to help pupils to understand how to apply the concepts and skills which are being learned and how to make use of them to solve problems. For many pupils this will require a great deal of discussion and oral work ……."
Cockcroft Report 1982

To develop the ability to solve problems alongside the ability to calculate, I propose the following strategies, and the sequence is significant:

1. The development of the mental tool kit should be coupled with using this tool kit to solve problems.
2. As problems become increasingly complex, and involve larger numbers, pupils should be encouraged to write down interim steps.
3. Pupils should be helped and encouraged to evaluate their efforts in terms of the principles of appropriateness, effectiveness and efficiency.
4. Teachers should build on pupils' work to develop efficient methods of calculation.

Efficient methods of calculating emerge from critically evaluating how pupils solve problems. Efficient methods of calculating are not a prerequisite to solving problems.

The line of progression is developing efficiency, based on the evidence of pupils' written work thus far. So, looking back at the examples of pupils' work above, the question is: "What does the teacher do next?"

Clearly, a discussion amongst the pupils, discussing different strategies used, can be productive. Pupils might be asked to explain how they saw the problem and what mental mathematical tools they considered using, and why. They may be asked to explain why the different approaches produced similar solutions to the problem. Teaching points will emerge, such as those related to subtraction being the inverse of addition, multiplication being the inverse of division, multiplication being repeated addition, and division being repeated subtraction.

They may be asked to judge what they thought were the 'best' methods, and why?

The teacher may choose to develop some of these points. For example, to the 'repeated subtractor', such as Claire:

In subtracting sixty twos, individually, she might have considered subtracting batches of sixty two. She could have used her understanding of multiplying sixty two by ten, to get 620 and mentally halved to find out that 62 x 5 = 310. She could have used her understanding of doubling and redoubling to find out that 62 x 4 = 124 x 2 = 248.
274 – 248 = 26.

Or, in the last question, in finding out how many thirty fours she can subtract from 760, she could have, initially subtracted in batches of 10. For example:

760		
340	-	10
420		
340	-	10
80		
34	-	1
46		
34	-	1
12		

I can subtract 22 thirty-fours from 760 (remainder 12)
I can make 22 bags of sweets and there will be 12 left over.

Or, even more efficiently:

760		
680	-	20
80		
68	-	2
12		

I can make 22 bags of sweets and there will be 12 left over.

You can see how the traditional algorithm for 'long division' can be developed from the work of pupils when they are using the strategy of repeated subtraction. The link between repeated subtraction and division needs to be demonstrated and explained. It may be justified in terms of its efficiency. The more stylised expression of this algorithm, which starts with the exhortation to find out how many thirty fours in seventy six, may be unhelpful, in that the style occludes the mathematics, and this fractures the link between what the pupil knows and understands, and an efficient way of recording a calculation. If you really feel that the stylised version is important, then it should start with the question 'how many tens of 34 can you take away from 760?" but this feels awkward to me.

To Yiu-Wai, 'the doubler', I think the traditional 'long division' algorithm is unnecessary. His method is efficient, though I might advise him to set out his working so that it is more reader-friendly.

34	1
68	2
136	4
272	8
544	16

I can take 16 lots away, (but an extra 8 lots is too much), another 4 lots, and another 2 lots.
544 + 136 + 68 = 748
I can make 22 bags of sweets and there are 12 left over.

How closely intertwined are division and multiplication. This connectivity needs to be made explicit. It is a shame that so many adults, including teachers, cite 'long division' as something they never understood.

To continue with efficient ways of recording multiplying:

To those who are using repeated addition, I would wish to draw their attention to Zoë's work in Problem 3, about the comics. Drawing on the mental tool kit, multiplying by powers of ten, and by single digit multiples of powers of ten, efficient methods can be developed.

In working out 357 x 24:

```
357 x 10 =      3570
357 x 10 =      3570
357 x  4 =      1428
____________________
357 x 24 =      8568
```

which may be expressed as:

```
357 x 20 =      7140
357 x  4 =      1428
____________________
357 x 24 =      8568
```

which may be expressed as:

```
 357
  24
____
7140
1428
____
8568
```

But who is to say if this is actually more efficient? All three pupils, in problem 3 above, know what they are doing and they get to an accurate solution quickly. And the stylised version of the last example often starts with "put a nought down and multiply by two" which carries with it a danger of unhooking it from something they understand and moving it into the realms of a 'trick', thus fracturing progression.

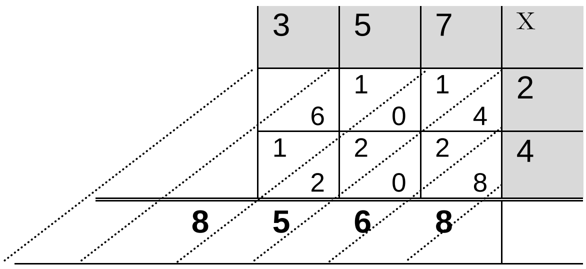
In other parts of the world there are alternative ways of efficiently recording multiplication, for example:

	300	**50**	**7**
20	6 000	1 000	140
4	1 200	200	28

6 000 + 1 000 + 140 + 1 200 + 200 + 28 = <u>8568</u>

I can see some strong connections between this method and the use of the place-value cards, mentioned in the previous chapter.

This presentation has itself become stylised, and some of you may recognise the similarity between the stylised method below and a device known as 'Napier's Bones'.



Notice how each cell is set out as the product of two digits. The sums along each 'diagonal' are found and recorded at the bottom, to give the solution 8568. It is an interesting investigation to see how this stylised version links with the presentation above.

For ‘ the doubler’, the stylised version is often known as “Russian Multiplication”:

“double”	“halve” (ignore remainder)	Is the column next to this odd or even?
~~357~~	24	even
~~714~~	12	even
~~1428~~	6	even
2856	3	odd
5712	1	odd
8568		

I cross out those numbers in the left-hand column, which are associated with an “even” in the right hand column, and then sum what is left. Again, finding out why this works is an interesting investigation. It links closely with Yiu-Wai’s own approach to multiplying. However, as a starting point for teaching a method, it may be too remote from pupils’ experience to hook onto their current understanding.

Stylised methods may well be adopted (and in some cases, adapted) by pupils, but this is the last stage in the process and not the first. They may arrive at this stage by being helped to see how their own recording may be made more efficient and being helped at every step to relate back to their own mental strategies and understanding. These stylised approaches are an end point to a long process and emerge out of the need to solve complex problems. They are not a **substitute** for working mentally, but depend on mental understanding so they may be used flexibly and with confidence.

I taught for some years in secondary schools and tried to train pupils in using stylised approaches in order to ‘patch over’ mental misconceptions. It does not work. The mental tool kit, even, at the last resort, without the stylised methods, can help pupils be better problem solvers.

I must make it clear, at this stage, that I am nervous about schools who declare themselves ‘traditional algorithm free zones’. This misses the point. The traditional algorithm, such as ‘subtraction by decomposition’ or ‘long division’ may well be understood as an efficient way of recording secure mental understanding. And pupils are entitled to be taught how to develop their own methods efficiently.

I quoted an example earlier, where a pupil was required to use a ‘written method’ to subtract seven from sixteen. Not only was this inefficient, but it was also inappropriate. Equipped with the mental tool kit, pupils should think carefully about each problem with which they are presented. Given an opportunity to think, they consider an appropriate method. An exhortation to “do all the following problems using the method I have just told you” can get in the way of pupils thinking for themselves, and lead to inappropriate ways of working. This applies equally to the use of the calculator.

15. APPROPRIATE CALCULATOR USE

Some examples of inappropriate and appropriate calculator use. Supporting imagery.

Allow me a second illustration of questionable practice. Sometimes a counter example can be helpful.

I was observing a lesson where pupils were calculating percentages of a quantity. One of the questions was to calculate 25% of 400. Pupils took out their calculator and typed in:

$$25 \div 100 \times 400 =$$

Not surprisingly, they saw that they now had 100 on their display, and wrote it down. I suggested to one or two pupils that this was an interesting result, but the interest did not register! They had no feel for the problem they were solving, no connecting of 25% with ¼, and no sense of halving 400 and halving it again. I would argue that, using a calculator, this exemplifies an inappropriate method. They had not considered using their mental tool kit.

I am equally concerned with pupils using calculators to check their mental calculations. Pupils are presented with a page of written calculations, asked to work them out mentally, and then to check them on a calculator. To me, this sends out contradictory messages on what constitutes appropriate calculator use. If the calculator is so reliable then why not use it in the first place. A better way of checking mental calculation might be to use inverse operations. For example, if I mentally calculate $76 \div 9$ to be 8 remainder 4, then $9 \times 8 + 4$ should return me to the 76.

A more appropriate relationship between calculator use and mental methods is where, in a complex calculation, pupils round the numbers to the nearest unit, or ten, or hundred, calculate mentally an approximate answer and then use this approximation to check the reasonableness of the calculator display.

One of the major outcomes of the 'CAN' (Calculator Aware Number) Project, was that it helped pupils develop mental strategies. Good calculator use is not about replacing the mental tool kit, but about enhancing its development.

And, of course, calculators are helpful, when working with realistic data, solving problems which involve numbers expressed to a large number of significant figures. Dividing a number with three digits by a number with one or two digits using a mix of mental and written methods is appropriate, but more complex calculations may more appropriately solved using a calculator.

In 1997, at an A.T.M. conference, I learnt how to use the soroban – the Japanese abacus. I later attended an exhibition of mental calculation given by two of the tutors of the soroban. They were able to perform the most awesome calculations mentally. I asked them how they managed to do it, and they replied that they could see sorobans in their mind and could mentally manipulate the counters on the rods. I was impressed. Later on I was confronted with a complex calculation. I closed my eyes and brought the electronic calculator into my mind. I pressed the keys, but …… to no avail. I began to understand both the usefulness of the electronic calculator, and its limitations.

Developing the mental tool kit is the first priority. This can be added to, by recording interim steps, when calculations require it, moving into written calculation. Beyond this, the calculator is appropriate.

However, the calculator can be used effectively as a teaching and learning tool in developing pupils' insights and understanding of the number system.

The use of the constant key on the calculator is particularly helpful when used in conjunction with exploring journeys around both the hundred square and the multiplication square. The constant key can also raise awareness of numbers which extend beyond the boundaries of these configurations. The calculator can be used productively to establish understanding of place value.

16. SOME PRACTICAL TEACHING AIDS

The following pages include some of the materials that have been used to provide images of the structure of numbers:

A. The number square for teaching the relationship between numbers and addition and subtraction to a hundred.

B. A multiplication frame for teaching multiplication tables.

C. The Gattegno tens chart for teaching counting and the relationship between numbers to a million.

D. The Gattegno tens chart for teaching multiplication and division by powers of ten.

E. A set of place-value cards for teaching the composition of numbers to a thousand.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

	1	2	3	4	5	6	7	8	9	10
X	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
X	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
X	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**
100	**200**	**300**	**400**	**500**	**600**	**700**	**800**	**900**

1 000	**2 000**	**3 000**	**4 000**	**5 000**	**6 000**	**7 000**	**8 000**	**9 000**
10 000	**20 000**	**30 000**	**40 000**	**50 000**	**60 000**	**70 000**	**80 000**	**90 000**
100 000	**200 000**	**300 000**	**400 000**	**500 000**	**600 000**	**700 000**	**800 000**	**900 000**

0.01	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.1	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**
100	**200**	**300**	**400**	**500**	**600**	**700**	**800**	**900**
1 000	**2 000**	**3 000**	**4 000**	**5 000**	**6 000**	**7 000**	**8 000**	**9 000**
10 000	**20 000**	**30 000**	**40 000**	**50 000**	**60 000**	**70 000**	**80 000**	**90 000**
100 000	**200 000**	**300 000**	**400 000**	**500 000**	**600 000**	**700 000**	**800 000**	**900 000**

1	0	6	0	1	6
2	0	7	0	2	7
3	0	8	0	3	8
4	0	9	0	4	9

PLACE VALUE CARDS

Cut out along the heavy lines

1	0	0	6	0	0
2	0	0	7	0	0
3	0	0	8	0	0
4	0	0	9	0	0
5	0	0	5	0	5

17. EPILOGUE

This book has focused exclusively on the acquisition and application of number skills. The recurring theme has been that a knowledge and understanding of the structure of number empowers the learner. Provide and teach useful images which illustrate number structure, and the development of calculating skills is enabled.

Number-sense is knowing where numbers live within the structure and being able to visualise transformations which relate to the various number operations. It is knowing that, say, 3.26 lives between 3 and 4, and actually lives closer to 3. It is knowing that 3.26 lives between 3.2 and 3.3, and actually lives closer to 3.3. Knowing number neighbours helps when working out approximate solutions to calculations so that precise calculations can be checked and verified.

Number sense is understanding the relationship between operations and using this to attack calculations with confidence. It is knowing that multiplication can be used to solve a 'division' problem.

Number sense is knowing the relationship between numbers and using this knowledge to calculate quickly and efficiently. It is knowing that multiplying by, say, 25 is equivalent to multiplying by a hundred and then dividing by four.

We owe it to pupils to let them into the world of number, with its perfect regularity and accessible logic. This is at the heart of teaching number.

APPENDIX 1

Some contact addresses

1 Further articles on teaching mental arithmetic and large range of ideas and materials for use in the classroom are available from:

The Association of Teachers of Mathematics
7 Shaftesbury Street
Derby
DE23 8YB

2 Large versions of the Gattegno Tens Chart and Multiplication Charts, with teachers' notes are available from:

Education Initiatives
Cardew Farm
Dalston
Carlisle
CA5 7JQ

APPENDIX 2

A Numeracy Audit

The fifteen items used in numeracy survey in North East Lincolnshire schools.
(see 'Acknowledgements' at the start of this book)

		No evidence	Low				High
1	What emphasis is put on **building** mental methods calculating? (not just testing)	0	1	2	3	4	5
2	To what extent do pupils **practise** mental calculations?	0	1	2	3	4	5
3	To what extent do written calculations build on pupils' **mental** understanding?	0	1	2	3	4	5
4	To what extent do **pupils** select the most appropriate method of calculation?	0	1	2	3	4	5
5	Are a **variety** of methods for solving numerical problems taught and used?	0	1	2	3	4	5
6	To what extent is there an emphasis in **planning and teaching** on using appropriate, effective and efficient methods of calculating?	0	1	2	3	4	5
7	Is the use of calculators **appropriate** to pupils' development of calculating skills?	0	1	2	3	4	5
8	Is the classroom environment **helpful** in developing pupils' confidence and understanding of the relationship between numbers (e.g. practical equipment, number lines & squares)?	0	1	2	3	4	5
9	To what extent is the **acquisition** of numerical skills associated with their **application** (e.g. describing & analysing situations and solving problems; measuring etc)?	0	1	2	3	4	5
10	Do pupils, where appropriate, **discuss** how they tackle and solve mathematical problems?	0	1	2	3	4	5
11	Are pupils, helped to **develop** their mathematical vocabulary (e.g. displays of key words etc)?	0	1	2	3	4	5
12	To what extent is there a secure **progression** in the development of number skills?	0	1	2	3	4	5
13	Is there evidence of **using** mathematical skills in **other** curriculum areas (e.g. measurement, calculation, handling data)?	0	1	2	3	4	5
14	Is the cross-curricular use of mathematical skills **planned** for?	0	1	2	3	4	5
15	Is there evidence of **teaching** mathematical skills in outside-school everyday contexts, i.e. functional numeracy (e.g. numbers on buses, telling the time, money matters, **interpreting** data & graphs etc)?	0	1	2	3	4	5