The Doomed Fly: a kinematic teaser

by Ian Tame

In this article we use a variety of analytical and graphical methods to determine the remaining lifespan of a doomed fly trapped between two moving walls.

Our scenario begins with two facing vertical walls 100 metres apart and the fly standing on one of the walls. At this instant the walls start to approach each another at a speed of 100 ms^{-1}. Simultaneously the fly takes-off with a constant horizontal speed of 200 ms^{-1} in the direction of the opposite approaching wall (see Figure 1).

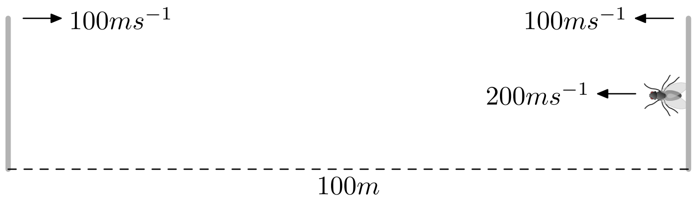

Abbildung 1: Initial Scenario

In modelling this situation, we will also make the following assumptions:

i) the fly reaches the speed of 200 ms^{-1} instantaneously from take-off; and

ii) during its subsequent flight, whenever the fly reaches an oncoming wall it instantaneously reverses its horizontal direction towards the other approaching wall and flies at a constant speed of 200 ms^{-1}.

The problem is to determine the duration and total length of the fly's flight, before the fly gets unceremoniously squashed between the two walls?

Method 1: a series approach

This approach breaks the fly's motion down into alternating and successive flights between the walls. As the fly's direction alternates, the horizontal distance between the walls decreases. So that the lengths of each of these flights can be summed to form a series whose infinite sum is the total horizontal distance flown by the fly (before it meets its demise in the vertical plane where the walls subsequently meet). Thereafter, with the total distance travelled known, the total time of the flight can be readily calculated.

For the sake of clarity we will denote the wall on which the fly initially stands by *Wall A*, and the other wall by *Wall B*.

During the first phase of the subsequent motion in which the fly flies horizontally from *Wall A* to *Wall B*, its speed (200 ms^{-1}) is twice that of the oncoming wall *Wall B* (100 ms^{-1}). Therefore during this phase it has traversed twice the horizontal distance of *Wall B*. And since the initial relative distance between the fly and *Wall B* was 100 metres, the fly must have flown $\frac{2}{3}(100)$ metres, during which time *Wall B* has moved $\frac{1}{3}(100)$ metres. Also observe *Wall A* has also moved towards *Wall B* a distance of $\frac{1}{3}(100)$ metres. Hence at the end of the first phase of motion, the distance between the walls has been reduced to $(100 - \frac{1}{3}(100) - \frac{1}{3}(100))$ metres $= \frac{1}{3}(100)$ metres (see Figure 2).

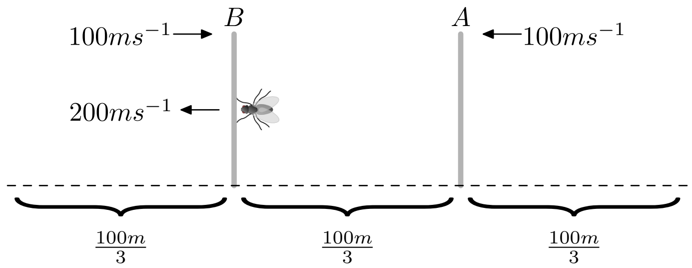

Abbildung 2: Scenario after Phase 1

In the second phase of motion the fly first returns from *Wall B* to *Wall A*. In analogy to the reasoning above for the first phase, the fly will have flown $\frac{2}{3}(\frac{1}{3}(100))$ metres and *Wall A* would have moved a further $\frac{1}{3}(\frac{1}{3}(100))$ metres. So by the end of the second phase of motion, the distance between the walls has been further reduced to $(\frac{1}{3}(100) - (\frac{1}{3})^2(100) - (\frac{1}{3})^2(100))$ metres $= (\frac{1}{3})^2(100)$ metres (see Figure 3).

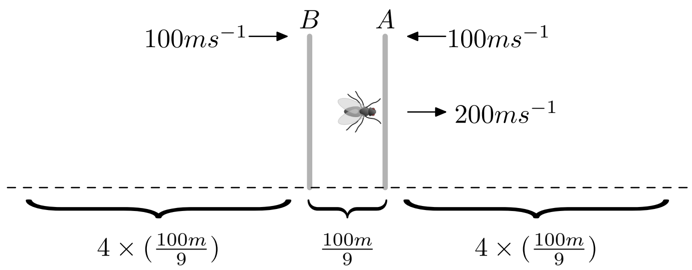

Abbildung 3: Scenario after Phase 2

Likewise in the third phase, in which the fly returns again from *Wall A* to *Wall B*, it will have flown $\frac{2}{3}((\frac{1}{3})^2(100))$ metres during which time *Wall B* would have moved $\frac{1}{3}((\frac{1}{3})^2(100))$ metres. And the subsequent distance between the walls yet further reduced to $((\frac{1}{3})^2(100) - (\frac{1}{3})^3(100) - (\frac{1}{3})^3(100))$ metres $= (\frac{1}{3})^3(100)$ metres.

Continuing indefinitely in this manner, the total distance flown by the fly S (metres), is the sum of the horizontal distances travelled in an infinite number of such phases. That is, S is given by the following infinite series:

$$\begin{aligned} S &= \Big(\frac{2}{3}\Big)(100) + \Big(\frac{2}{3}\Big)\Big(\frac{1}{3}\Big)(100) + \Big(\frac{2}{3}\Big)\Big(\frac{1}{3}\Big)^2(100) + \Big(\frac{2}{3}\Big)\Big(\frac{1}{3}\Big)^3(100) + \ldots \\ &= \Big(\frac{2}{3}\Big)(100)\Big(1 + \frac{1}{3} + \Big(\frac{1}{3}\Big)^2 + \Big(\frac{1}{3}\Big)^3 + \ldots\Big) \\ &= \frac{2}{3}(100)\Big(\frac{1}{1-\frac{1}{3}}\Big) \\ &= \frac{2}{3}(100) \times \frac{3}{2} \\ &= 100 \text{ metres.} \end{aligned}$$

Then since the speed of the fly was constant ($200\ ms^{-1}$) throughout its motion, it is a trivial step to determine the total duration T (secs) of the fly's flight using

$$T = \frac{distance}{speed} = \frac{100}{200} = \frac{1}{2} \text{ sec.}$$

Method 2: a recursive sequence approach

An alternative approach is to define the wall separation distance at the start of phase n by S_n metres and then construct the recurrence sequence for S_n $(n \geq 1)$. In this manner, we obtain:

$$\begin{aligned} S_1 &= \frac{2}{3}(100) \\ S_n &= \frac{1}{3}S_{n-1} \quad (n \geq 2). \end{aligned}$$

From this relation, we deduce that

$$S_n = \Big(\frac{1}{3}\Big)^{n-1} S_1.$$

From which we obtain (in analogy to the series approach above), the total distance S flown by the fly is

$$S = \sum_{i=1}^{n} S_i = S_1 \left(1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \ \ldots \right)$$
$$= 100 \text{ metres.}$$

It is also instructive to observe that we can also construct a recurrence relation for the time of flight T_n $(n \geq 1)$ secs taken by the fly at phase n. By which we obtain:

$$T_1 = \frac{S_1}{200} = \frac{\frac{2}{3}(100)}{200} = \frac{1}{3} \quad \left(\text{since } \mathit{time} = \frac{\mathit{distance}}{\mathit{speed}}\right)$$
$$T_n = \frac{1}{3} T_{n-1} \quad (n \geq 2) \quad \left(\text{since for each } n,\ T_n \propto S_n \text{ and } S_n = \frac{1}{3} S_{n-1}\right).$$

Hence it follows that

$$T_n = \left(\frac{1}{3}\right)^{n-1} T_1,$$

from which (in analogy to the S_n series sum above), the total duration of the flight of the fly T is

$$T = \sum_{i=1}^{n} T_i = T_1 \left(1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \ \ldots \right)$$
$$= \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \ldots$$
$$= \frac{\frac{1}{3}}{1 - \frac{1}{3}}$$
$$= \frac{1}{2} \text{ sec.}$$

The velocity-time graph in Figure 4 below gives a pictorial representation of this method, and explicitly depicts the first four phases of the fly's flight. Observe that the distances flown by the fly in each of the first four phases, can be readily determined by the respective areas S_1, S_2, S_3, S_4 shown on the graph.

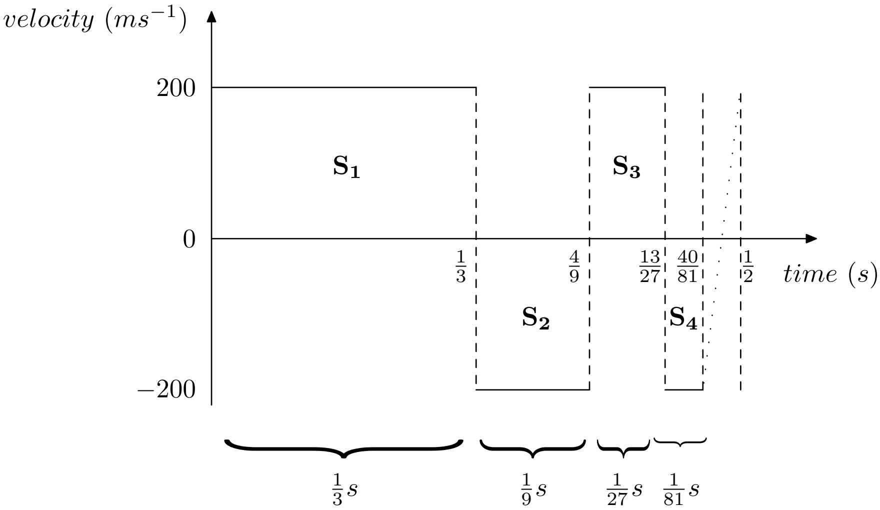

Abbildung 4: velocity-time graph of the fly's flight

Method 3: a grass-roots approach

There is, however, a more dynamical grass-roots approach to this problem, which also provides the most efficient and elegant solution!

For this method we just need to use the standard speed-distance-time formula (for constant speed). For although clearly the speeds of each of the respective walls are constant (and in fact the same, viz: 100 ms^{-1}); we also make the subtle observation that throughout the motion the speed of the fly is constant, viz: 200 ms^{-1}. Do not confuse this with the fly's velocity vector which changes each time the fly reverses its direction when it reaches an oncoming wall. So in essence we just concern ourselves with the scalar quantities and consider simultaneously the entire motion of the fly and the walls.

Walls: Each wall approaches the other with the constant speed 100 ms^{-1}. So they will meet at the mid-point of their initial separation, ie. after they have both moved $\frac{100}{2} = 50$ metres.

Fly: Since the fly's speed at 200 ms^{-1}, is double that of each of the walls, it would have travelled twice the distance traversed by each of the walls in the entire motion, viz: $2 \times 50m = 100$ metres.

These relationships can be utilized in speed-distance-time formula applied simultaneously to the entire motion of a wall and the fly, according to:

$$Time = \frac{Distance_{\texttt{WALL}}}{Speed_{\texttt{WALL}}} = \frac{Distance_{\texttt{FLY}}}{Speed_{\texttt{FLY}}}$$

That is

$$Time = \frac{50}{100} = \frac{Distance_{\texttt{FLY}}}{200}.$$

Giving the total distance flown by the fly ($Distance_{\texttt{FLY}}$) as $100m$ and the total duration ($Time$) of the flight of the fly as $\frac{1}{2}$ sec.

The methodology utilized in this article is very much in the vein of problem solving and mathematical modelling approaches elucidated in university preparatory mathematics texts. Typical examples include: Griesler [1], Schichl u. Steinbauer [2] and Tao [3].

Obvious extensions to this teaser include the investigation of scenarios in which only one wall moves towards the other, or a more general derivation of distance and time formulae for the fly when the initial distance between the walls and the given velocities are expressed as parameters.

Literatur

[1] D. Grieser: *Mathematisches Problemlösen und Beweisen - eine Entdeckungsreise in die Mathematik*. Springer Spektrum, Wiesbaden, 2013.

[2] H. Schichl, R. Steinbauer: *Einführung in das mathematische Arbeiten*. Springer Spektrum, Berlin Heidelberg, 2012.

[3] T.Tao: *Solving Mathematical Problems*. OUP, New York, 2006.